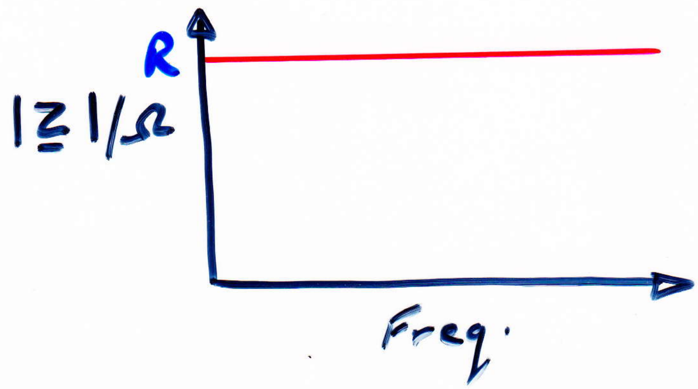
Frequency Response Analysis

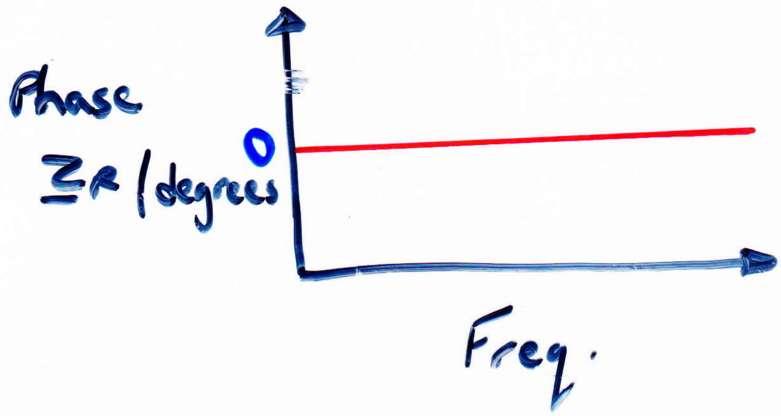
Let us consider in turn

- resistor
- inductor
- capacitor

Resistor

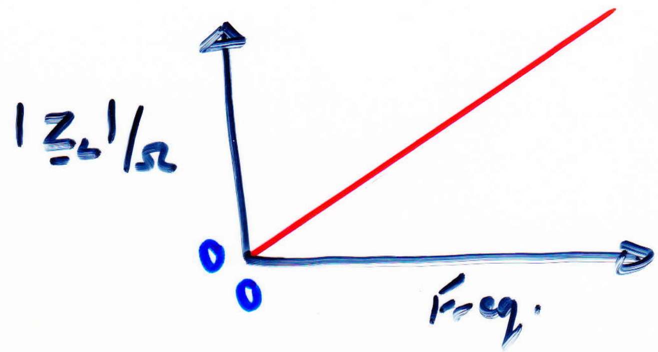
$Z_R = R = R\angle 0^\circ$

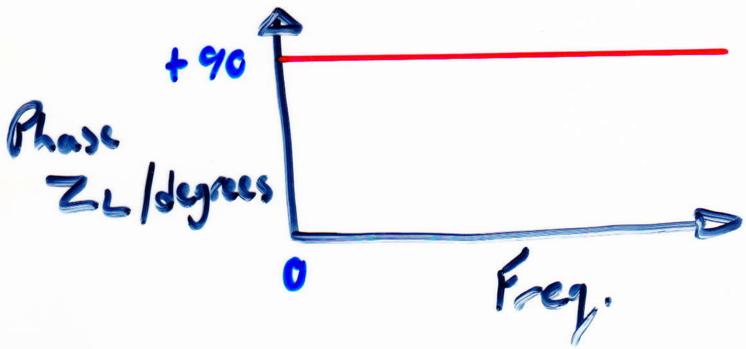




Inductor

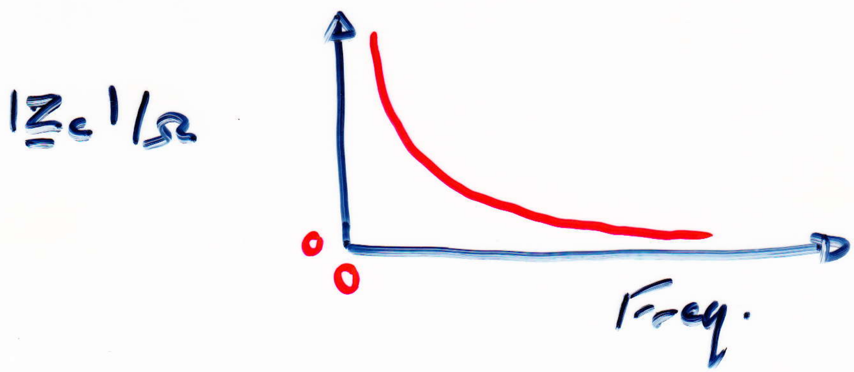
$$\underline{Z}_L = j\omega L = \omega L \angle 90^\circ$$

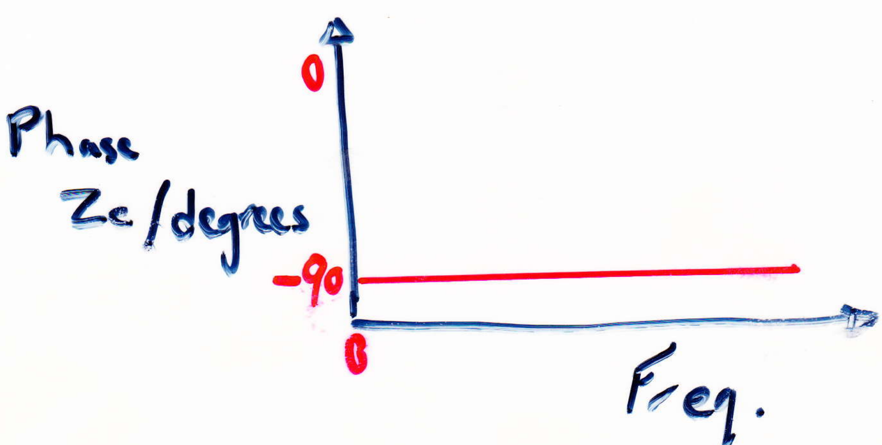




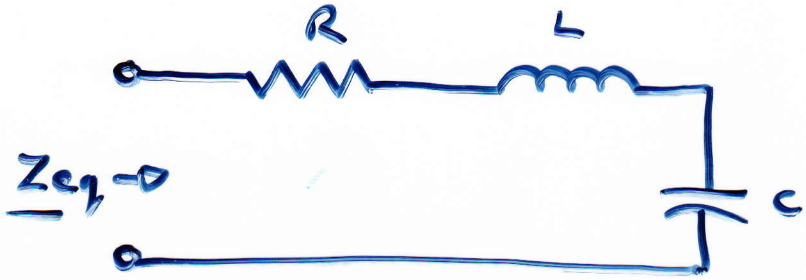
Capacitor

$$\underline{Z}_C = \frac{1}{j\omega C} = \frac{1}{\omega C} \angle -90^\circ$$



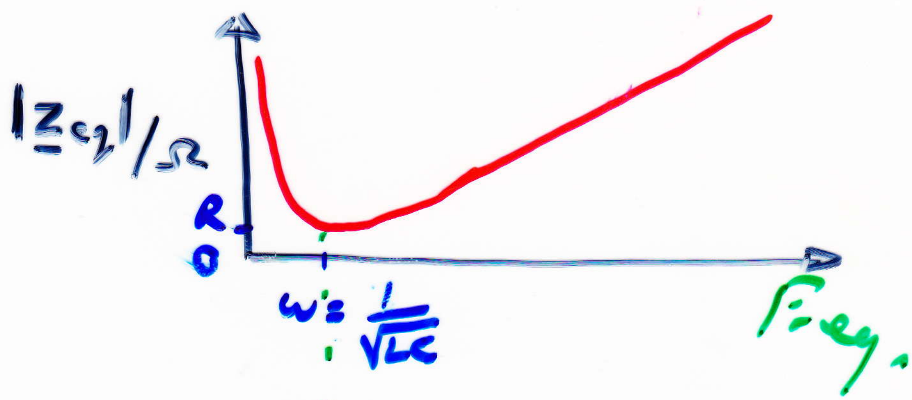


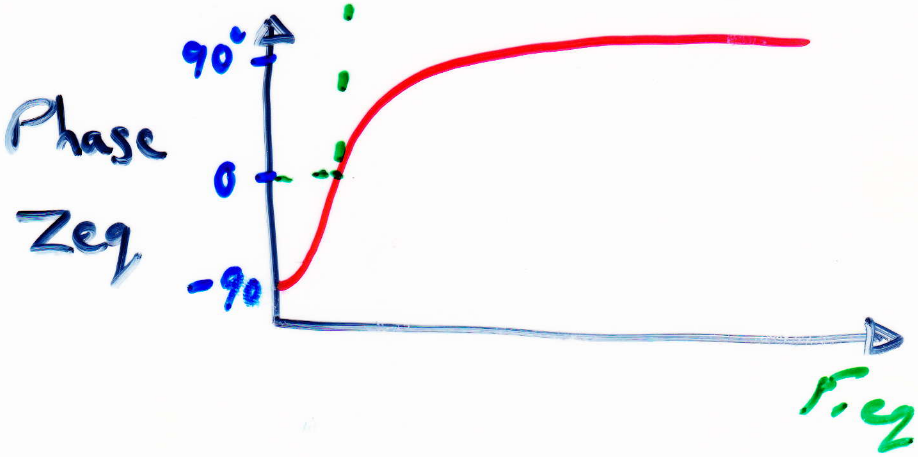
Consider RLC in series



$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C}$$

$$Z_{eq} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C}$$





With

$$\underline{Z}_{eq} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C}$$

Let us replace $j\omega$ with s

$$\therefore \underline{Z}_{eq} = \frac{s^2 LC + sRC + 1}{sC}$$

This ratio of two polynomials can be generalised to

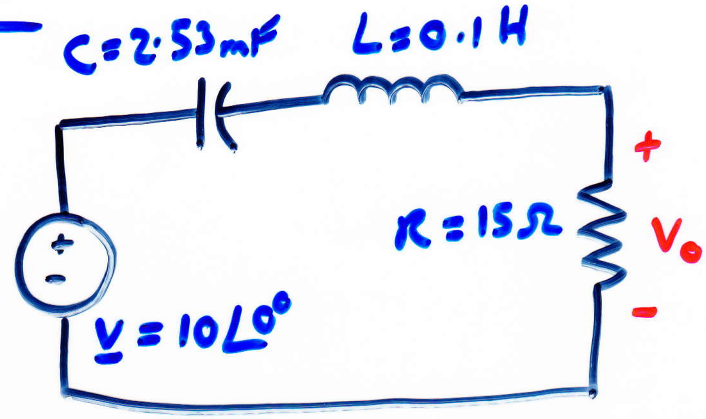
$$\underline{Z}(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

This general form also holds for

- voltages
- currents
- admittances
- gains

of networks.

Example



To find V_o use voltage division.

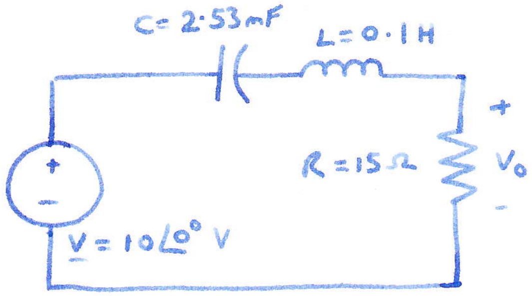
$$\underline{V}_o = \left(\frac{R}{R + j\omega L + \frac{1}{j\omega C}} \right) \underline{V}_s$$

$$\underline{V}_o = \left(\frac{j\omega CR}{(j\omega)^2 LC + j\omega CR + 1} \right) \underline{V}_s$$

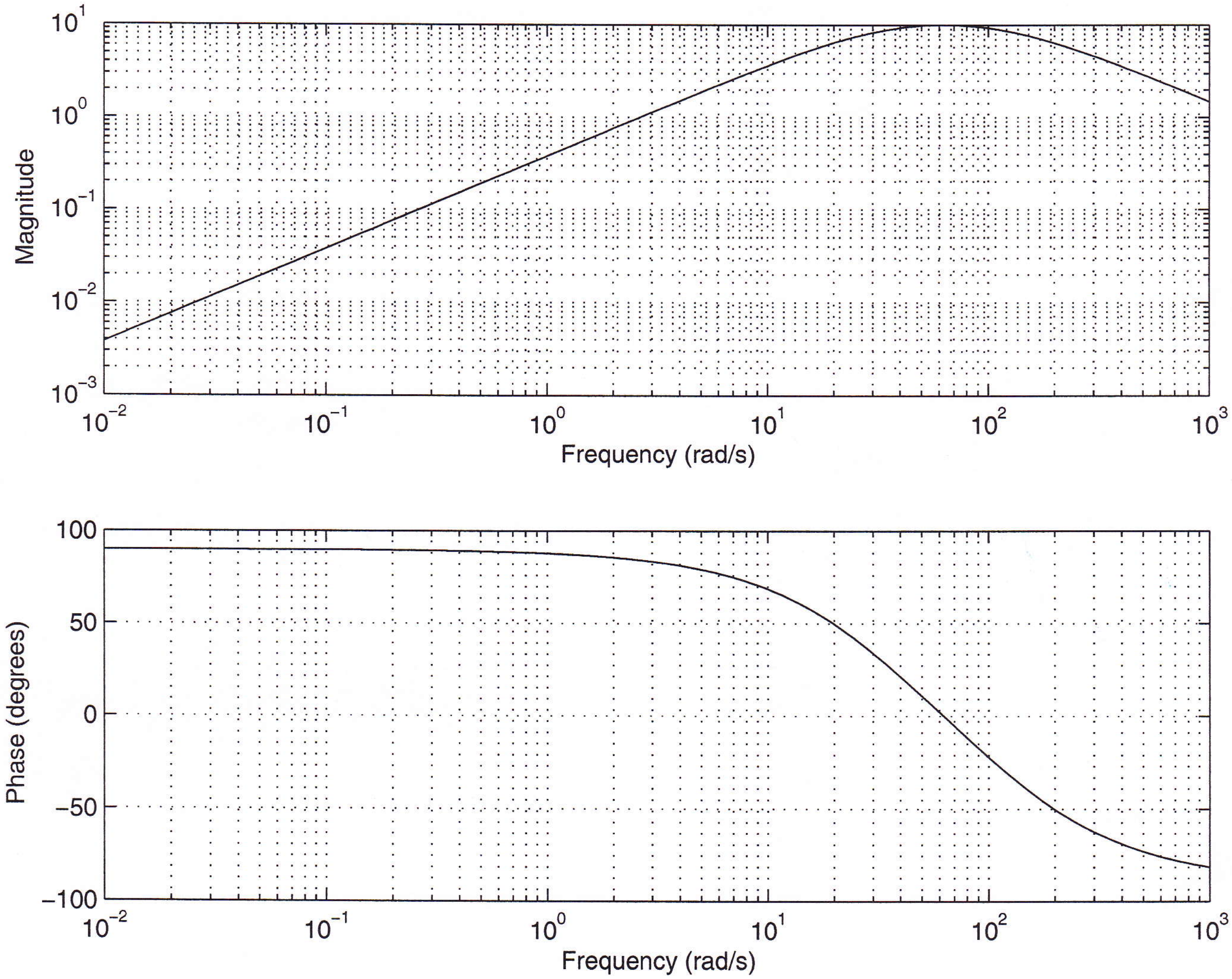
$$V_o = \left(\frac{(j\omega)(37.95 \times 10^{-3})}{(j\omega)^2 (2.53 \times 10^{-4}) + j\omega(37.95 \times 10^{-3}) + 1} \right) 10\angle 0^\circ$$

$$V_o = \frac{S\, 37.95 \times 10^{-2}}{S^2\, 2.53 \times 10^{-4} + S\, 37.95 \times 10^{-3} + 1}$$

freqs(b,a).



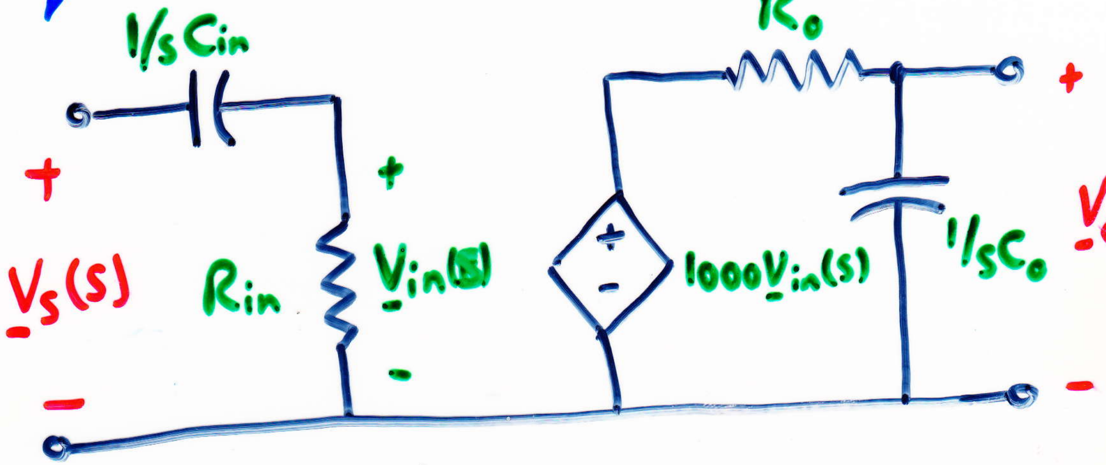
$$\underline{V_o} = \frac{s\, 37.95 \times 10^{-2}}{s^2\, 2.53\times10^{-4} + s\, 37.95\times10^{-3} + 1}$$



Example

Gain of an audio amplifier

Equivalent circuit

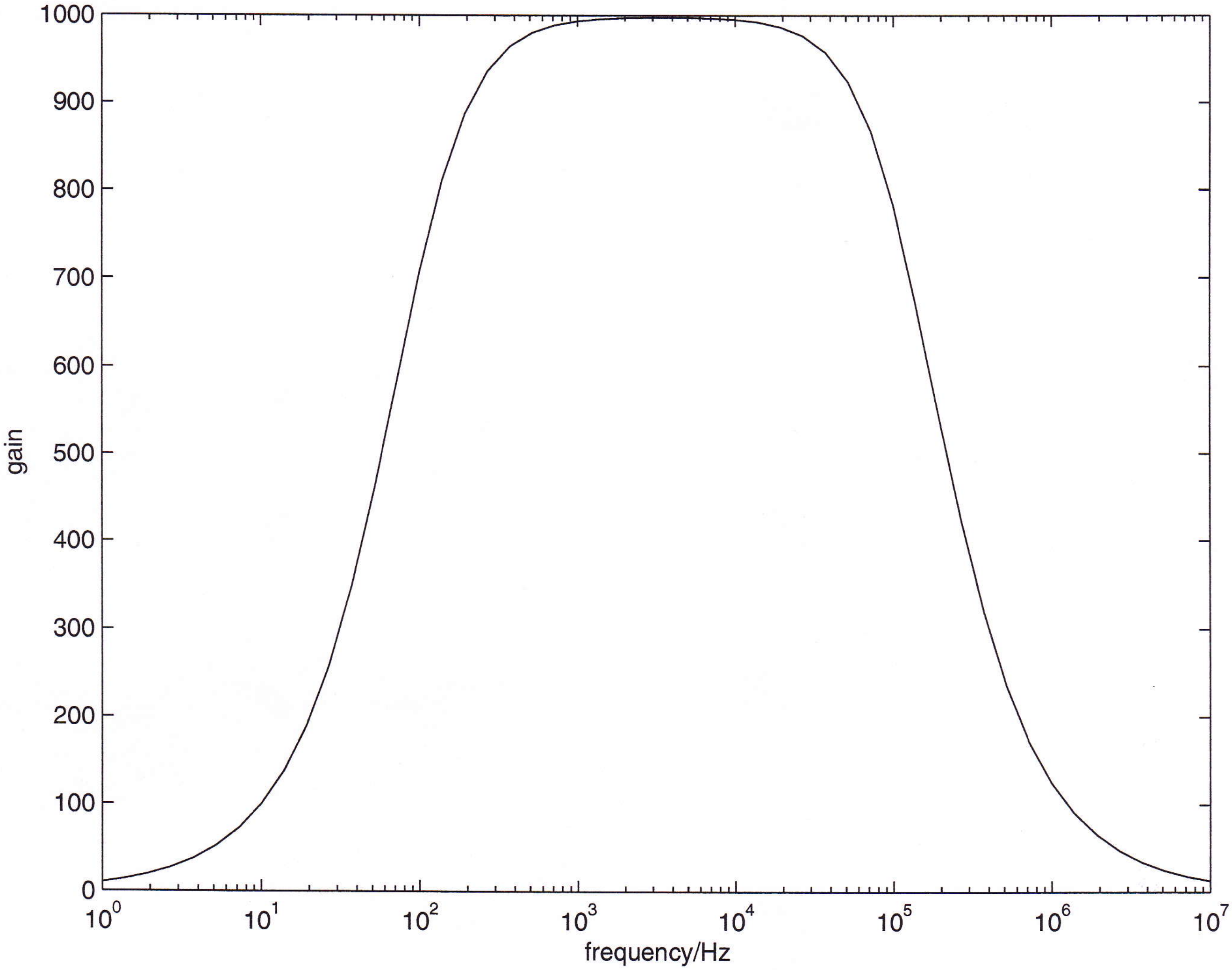


$$\underline{G}_v(s) = \frac{\underline{V}_o(s)}{\underline{V}_s(s)}$$

Voltage division

$$\underline{G}_v(s) = \frac{\underline{V}_o(s)}{\underline{V}_s(s)} = \frac{\underline{V}_{in}(s)}{\underline{V}_s(s)}\,\frac{\underline{V}_o(s)}{\underline{V}_{in}(s)} = \left[\frac{R_{in}}{R_{in} + 1/sC_{in}}\right](1000)\left[\frac{1/sC_o}{R_o + 1/sC_o}\right]$$

$$\underline{G}_v(s) = \left[\frac{sC_{in}R_{in}}{1 + sC_{in}R_{in}}\right](1000)\left[\frac{1}{1 + sC_oR_o}\right]$$


1000
900
800
700
600
500
400
300
200
100
0
gain
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
frequency/Hz

Sinusoidal Frequency Analysis

Bode Plots

Poles & Zeros

We have seen that the network transfer function can be expressed as the ratio of two polynomials.

$$\underline{H}(s) = \frac{\underline{N}(s)}{\underline{D}(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0} \qquad (a)$$

Note: transfer functions are

Input	Output	Transfer Function	Symbol
voltage	voltage	voltage gain	$\underline{G}_v(s)$
current	voltage	transimpedance	$\underline{Z}(s)$
current	current	current gain	$\underline{G}_i(s)$
voltage	current	transadmittance	$\underline{Y}(s)$

Now (a) can be written in form

$$\underline{H}(s) = \frac{K_0 (s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

(e.g. $s^2-5s+6 = (s-2)(s-3)$)

K_0 - constant
$z_1, \ldots, z_m$ are roots of $N(s)$
$p_1, \ldots, p_n$ are roots of $D(s)$

If $s = z_1$ or $z_2, \ldots, z_m$ $\underline{H}(s)$ becomes zero **ZEROS**

If $s = p_1$ or $p_2, \ldots, p_n$ $\underline{H}(s)$ becomes ∞ **POLES**

Zeros & poles can be complex, however, they must occur in conjugate pairs.

E.g.

$$G(s) = \left[\frac{s}{s+100\pi}\right] 1000 \left[\frac{40{,}000\pi}{s+40{,}000\pi}\right]$$

$\therefore$ $K_0 = 4\times10^7\pi$

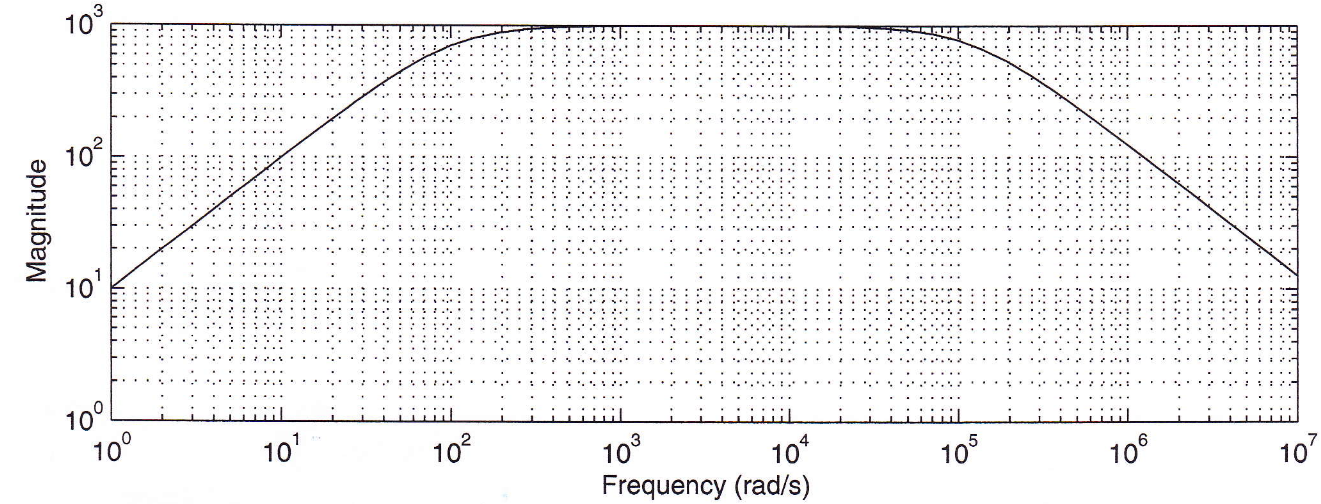
$z_1 = 0$

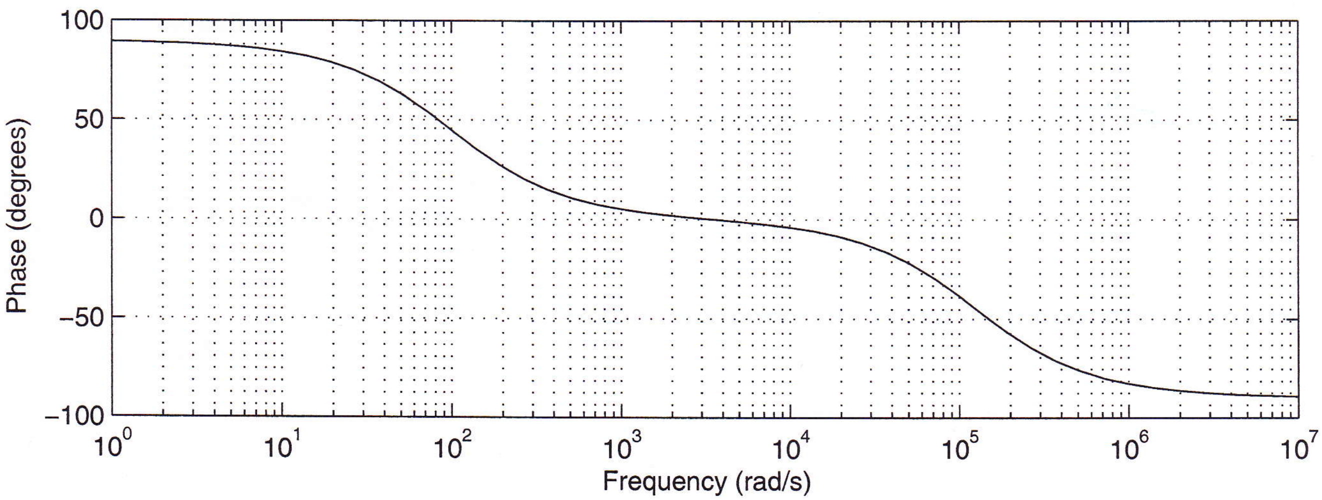
$p_1 = -100\pi$, $p_2 = -40{,}000\pi$

Putting into freq., $p_1 = -50$ Hz, $p_2 = -20{,}000$ Hz

$R_{in} = 1M\Omega$, $C_{in} = 3.18nF$, $R_o = 100\Omega$, $C_o = 79.58nF$

$$\underline{G}_v(s) = \left[\frac{s}{s + 100\pi}\right](1000)\left[\frac{40{,}000\pi}{s + 40{,}000\pi}\right]$$





Frequency Analysis.

Can express transfer function as

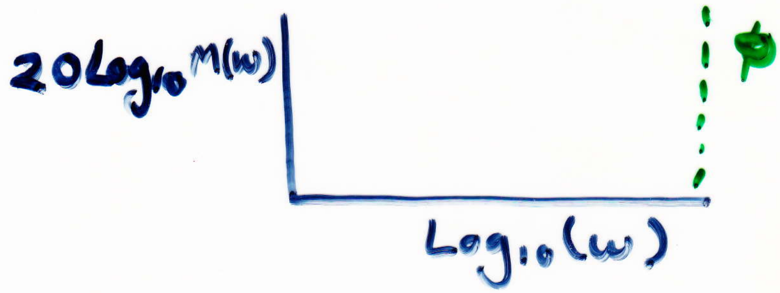
$$\underline{H}(jw) = \underbrace{M(w)}_{\text{Magnitude}} e^{j\overbrace{\phi(w)}^{\text{Phase.}}}$$

Aim is to examine magnitude & phase as functions of w.

Bode Plot

Tool for analysis and design of frequency dependent systems, filters, tuners & amplifiers. Rather than plotting the characteristic point by point can use straight-line approximations.

Axes :-



Use <u>decibels</u>

$$dB = 10 \operatorname{Log}_{10} \frac{\underline{P}_2}{\underline{P}_1}$$

P_n is a power

Using $\underline{P} = \underline{I}\,\underline{V}$ and Ohm's law can see have

$$dB = 10 \operatorname{Log}_{10} \frac{|\underline{V}_2|^2/R}{|\underline{V}_1|^2/R} = 10 \operatorname{Log}_{10} \frac{|\underline{I}_2|^2 R}{|\underline{I}_1|^2 R}$$

$$= 20 \operatorname{Log}_{10} \frac{|\underline{V}_2|}{|\underline{V}_1|} = 20 \operatorname{Log}_{10} \frac{|\underline{I}_2|}{|\underline{I}_1|}$$

Returning to our transfer function. Use general expression

$$\underline{H}(j\omega) = \frac{K_o (j\omega)^{\pm N} (1 + j\omega\tau_1)\left[1 + 2\zeta_3 (j\omega\tau_3) + (j\omega\tau_3)^2\right]\cdots}{(1 + j\omega\tau_a)\left[1 + 2\zeta_b (j\omega\tau_b) + (j\omega\tau_b)^2\right]\cdots} \quad (b)$$

Have the following factors

1. Freq-independent factor $K_o > 0$
2. Poles $(j\omega)^{-N}$, zeros $(j\omega)^{+N}$ at origin
3. Poles or zeros of form $(1 + j\omega\tau)$
4. Quadratic poles or zeros of form $1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2$

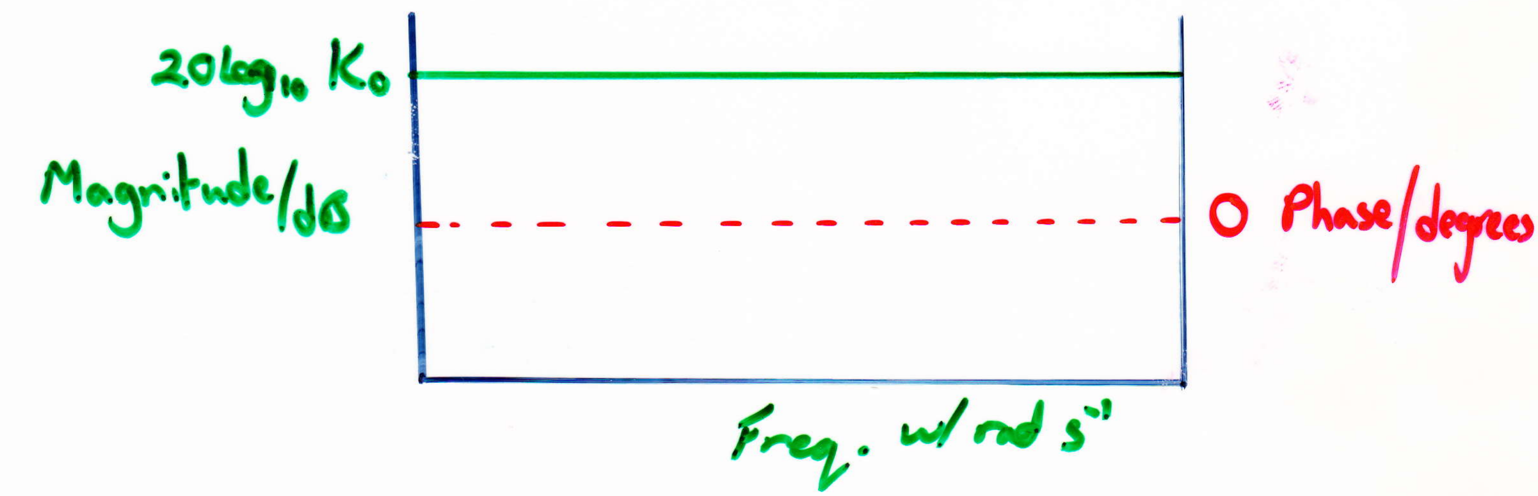
Manipulating (b)

$$20\,\mathrm{Log}_{10}|\underline{H}(j\omega)| = 20\,\mathrm{Log}_{10} K_o \pm 20N\,\mathrm{Log}_{10}|j\omega|$$
$$+ 20\,\mathrm{Log}_{10}|1 + j\omega\tau_1|$$
$$+ 20\,\mathrm{Log}_{10}|1 + 2\zeta_3(j\omega\tau_3) + (j\omega\tau_3)^2|$$
$$+ \ldots - 20\,\mathrm{Log}_{10}|1 + j\omega\tau_a|$$
$$- 20\,\mathrm{Log}_{10}|1 + 2\zeta_b(j\omega\tau_b) + (j\omega\tau_b)^2| \ldots$$

$$\angle\underline{H}(j\omega) = 0 \pm N(90^\circ) + \tan^{-1}\omega\tau_1 + \tan^{-1}\left(\frac{2\zeta_3\omega\tau_3}{1 - \omega^2\tau_3^2}\right)$$
$$+ \ldots - \tan^{-1}\omega\tau_a - \tan^{-1}\left(\frac{2\zeta_b\omega\tau_b}{1 - \omega^2\tau_b^2}\right) \ldots$$

Considering the individual factors.

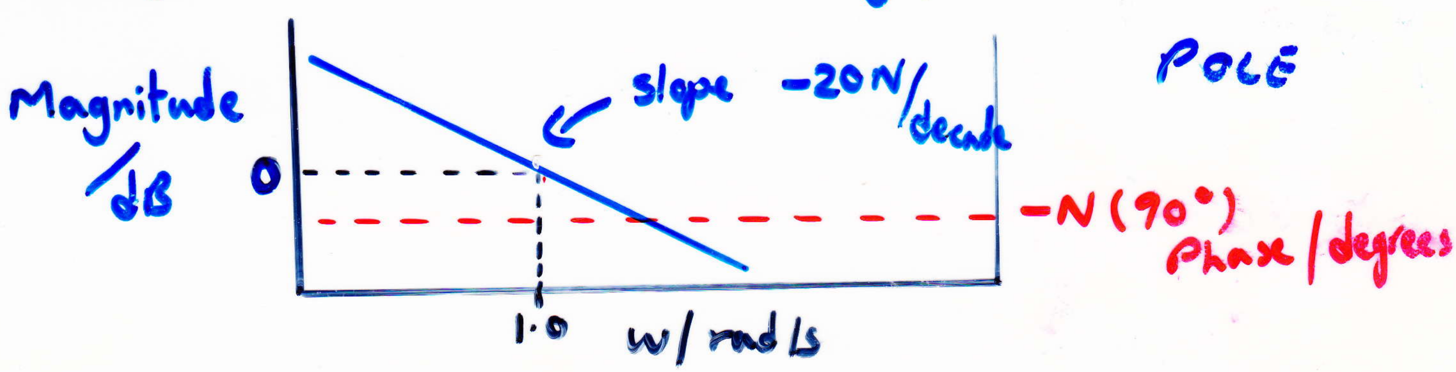
Constant Term



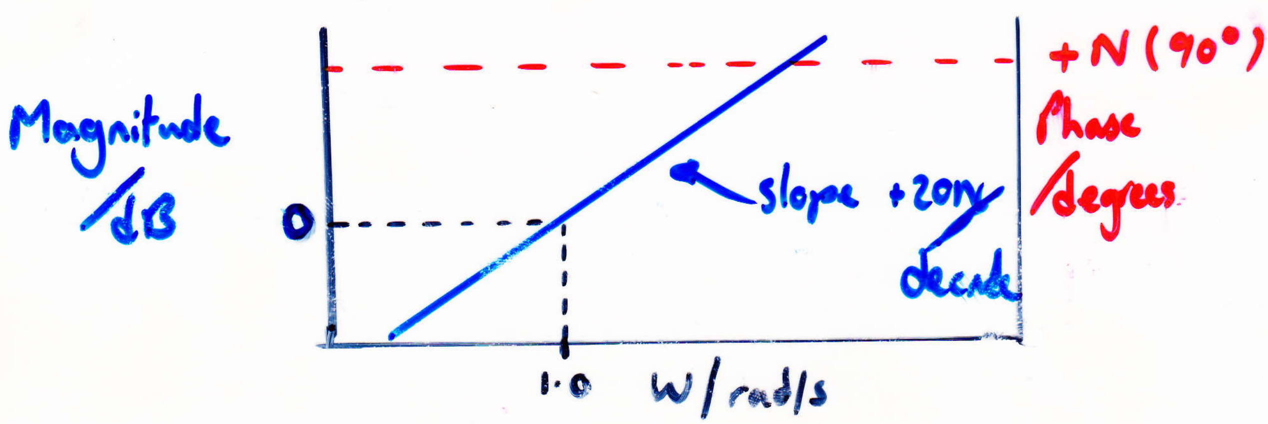
Poles or Zeros at the origin

$$(jw)^{\pm N} \longrightarrow \pm 20N \log_{10} w$$

POLE



ZERO



Simple Pole or Zero.

$$20 \log_{10} |1 + j\omega\tau_1|$$

For $\omega\tau \ll 1$ then $(1 + j\omega\tau) \simeq 1 \rightarrow 0\,\text{dB}$

For $\omega\tau \gg 1$ then $(1 + j\omega\tau) \simeq j\omega\tau \rightarrow 20 \log_{10} \omega\tau\,\text{dB}$

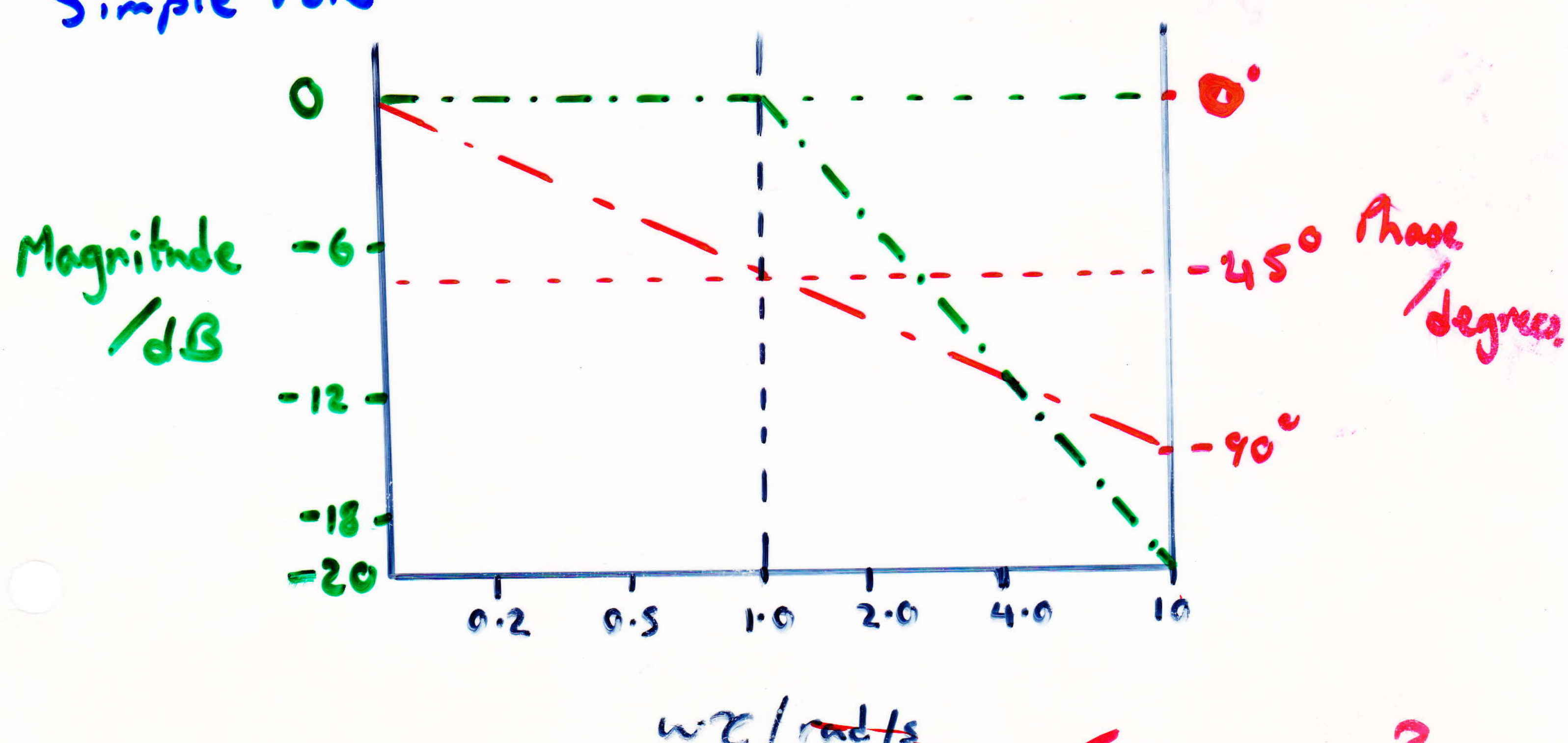
$\omega\tau = 1$ <u>Break</u> <u>Frequency</u> $20\log_{10}|1+j1|$

$= 20 \log_{10} \sqrt{2} = 3\,\text{dB}.$

$\phi = \tan^{-1} \omega\tau \quad \therefore \phi = 45^\circ$ at break freq.

ϕ varies between 0° & 90°
At twice break freq $\phi = 63.4^\circ$.
At half break freq $\phi = 26.6^\circ$.

Simple Pole



For zero?