Sinusoids, sinusoidal forcing functions & phasors.

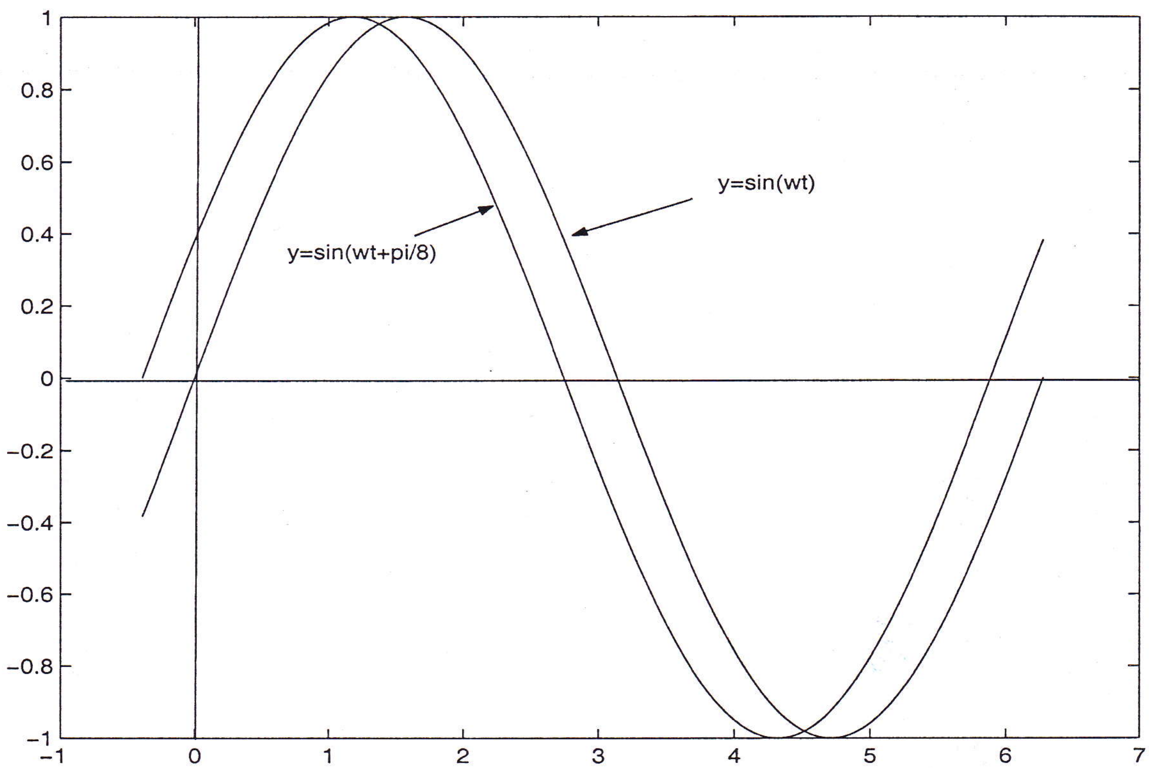
Introduction

So far considered circuit response to fixed value inputs or step inputs.
Now shall consider response to sinusoidal input or forcing function.

A number of approaches will be looked at, finishing with the phasor notation.

Sinusoids

Read section 7.1 - revision.



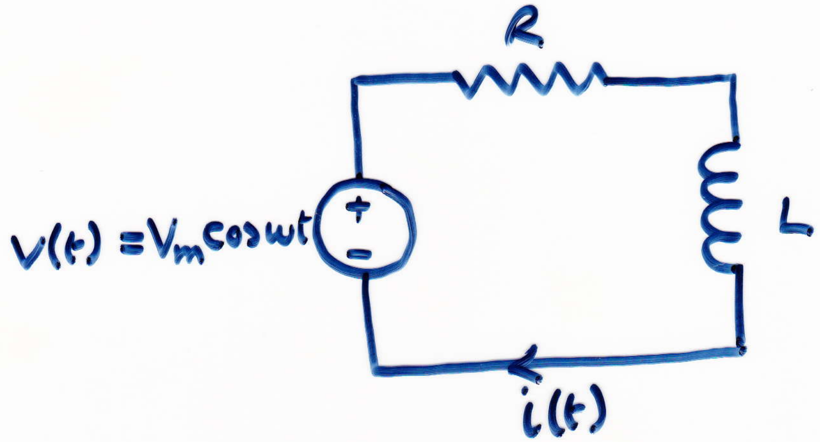
- $\omega = \frac{2\pi}{T} = 2\pi f$
- $cos\omega t = sin(\omega t + \frac{\pi}{2}),\ \ sin\omega t = cos(\omega t - \frac{\pi}{2})$

Sinusoidal Forcing Function.

Circuit source sinusoidal

$\Rightarrow$ Output response?

Consider a simple circuit.



From KVL

$$L\frac{di(t)}{dt} + Ri(t) = V_m \cos \omega t \qquad (1)$$

Assume forced response of circuit current $i(t)$ is of the form

$$i(t) = A\cos(\omega t + \phi)$$

or $i(t) = A\cos\phi\cos\omega t - A\sin\phi\sin\omega t$

$$= A_1\cos\omega t + A_2\sin\omega t \qquad (2)$$

$\therefore$ substituting into (1)

$$L\frac{d}{dt}(A_1 \cos wt + A_2 \sin wt) + R(A_1 \cos wt + A_2 \sin wt) = V_m \cos wt$$

$$-A_1 wL \sin wt + A_2 wL \cos wt + RA_1 \cos wt + RA_2 \sin wt = V_m \cos wt.$$

Equating coeffs of sine and cosines

$$-A_1 wL + A_2 R = 0$$

$$A_1 R + A_2 wL = V_m$$

$$\therefore \quad A_1 = \frac{RV_m}{R^2 + w^2L^2}$$

$$A_2 = \frac{wLV_m}{R^2 + w^2L^2}$$

$$\therefore \quad i(t) = \frac{RV_m}{R^2+w^2L^2} \cos wt + \frac{wLV_m}{R^2+w^2L^2} \sin wt$$

Comparing with (2) we have

$$A\cos\phi = \frac{R V_m}{R^2 + w^2 L^2}$$

$$A\sin\phi = \frac{-wL V_m}{R^2 + w^2 L^2}$$

So $\tan\phi = \dfrac{A\sin\phi}{A\cos\phi} = -\dfrac{wL}{R}$

Since

$$(A\cos\phi)^2 + (A\sin\phi)^2 = A^2(\cos^2\phi + \sin^2\phi) = A^2$$

$$A^2 = \frac{R^2 V_m^2}{(R^2 + w^2L^2)^2} + \frac{(wL)^2 V_m^2}{(R^2 + w^2L^2)^2}$$

$$= \frac{V_m^2}{R^2 + w^2 L^2}$$

So $A = \dfrac{V_m}{\sqrt{R^2 + w^2 L^2}}$

Using $i(t) = A\cos(\omega t + \phi)$ we have:

$$i(t) = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

Insight from inspection

$\phi = 0$ if $L = 0$

$R = 0$ if $\phi = -90^\circ$

If L & R present current lags the voltage by between 0 to 90°.

Consider using complex numbers:

$$e^{j\omega t} = \cos \omega t + j\sin \omega t$$

So input (in the simple RL circuit) becomes:

$$v(t) = V_m e^{j\omega t}$$

Expect response of form $i(t) = I_m e^{j(\omega t + \phi)}$

Working through example again

$$R I_m e^{j(\omega t + \phi)} + L \frac{d}{dt}\left(I_m e^{j(\omega t + \phi)}\right) = V_m e^{j\omega t}$$

$$R I_m e^{j(\omega t + \phi)} + j\omega L I_m e^{j(\omega t + \phi)} = V_m e^{j\omega t}$$

$$R I_m e^{j\phi} + j\omega L I_m e^{j\phi} = V_m$$

$$\therefore I_m e^{j\phi} = \frac{V_m}{R + j\omega L}$$

Recall $x + jy = re^{j\phi}$

$$r = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} y/x$$

So $$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2+w^2L^2}} e^{j\left[-\tan^{-1}\left(\frac{wL}{R}\right)\right]}$$

Use form $i(t) = I_m \cos(wt+\phi)$

then $$i(t) = \frac{V_m}{\sqrt{R^2+w^2L^2}} \cos\left[wt - \tan^{-1}\left(\frac{wL}{R}\right)\right]$$

Phasors

Now look at another notation.

$$v(t) = V_m \cos(wt+\phi) = Re\left[V_m e^{j(wt+\phi)}\right]$$

or as a complex number

$$v(t) = Re\left(V_m \angle\phi \, e^{jwt}\right)$$

Can drop the $Re(\cdot)$ and e^{jwt} and work with $V_m \angle\phi$

$$\therefore v(t) = V_m \cos(wt+\phi) = Re\left[V_m e^{j(wt+\phi)}\right] \equiv \underline{V} = V_m \angle\phi$$

Correspondingly

$$i(t) = I_m \cos(wt + \phi) = Re\left[I_m e^{j(wt+\phi)}\right]$$

$$\equiv \underline{I} = I_m \angle\phi$$

Using this notation consider the RL circuit again.

$$L\frac{di(t)}{dt} + Ri(t) = V_m \cos wt$$

Use $\underline{V}e^{jwt}$ where $\underline{V} = V_m \angle 0°$

So we have $L\dfrac{d\underline{I}e^{jwt}}{dt} + R\underline{I}e^{jwt} = \underline{V}e^{jwt}$

$$jwL\underline{I}e^{jwt} + R\underline{I}e^{jwt} = \underline{V}e^{jwt}$$

$$jwL\underline{I} + R\underline{I} = \underline{V}$$

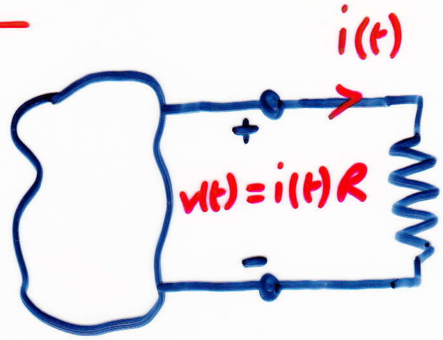
$$\therefore \underline{I} = \frac{\underline{V}}{R + jwL} = I_m \angle\phi = \underline{\frac{V_m}{\sqrt{R^2 + w^2L^2}}}\ \underline{\angle -\tan^{-1}\frac{wL}{R}}$$

$$\therefore i(t) = \underline{\frac{V_m}{\sqrt{R^2 + w^2L^2}}} \cos\left(wt - \underline{\tan^{-1}\frac{wL}{R}}\right)$$

Phasors applied to R, L & C.

Shall consider each type of component individually.

Resistor



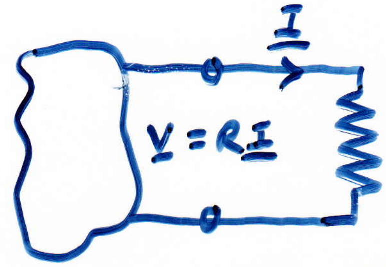
Applying the complex voltage $V_m e^{j(\omega t + \theta_v)}$

$$V_m e^{j(\omega t + \theta_v)} = R I_m e^{j(\omega t + \theta_i)}$$

$$V_m e^{j\theta_v} = R I_m e^{j\theta_i}$$

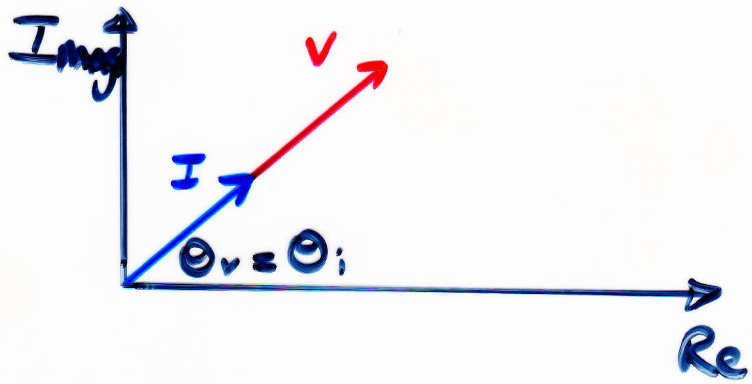
In phasor form

$$\underline{V} = R\,\underline{I}$$



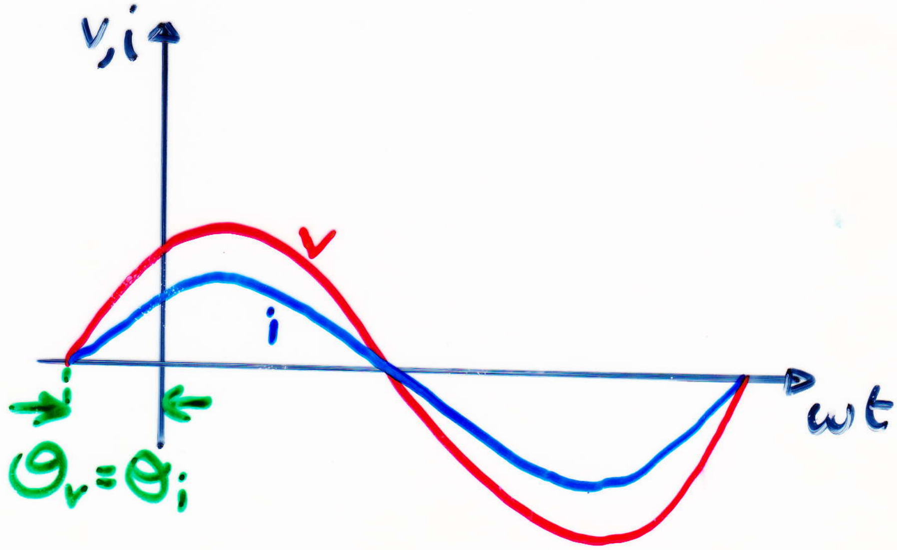
where $\underline{V} = V_m e^{j\theta_v} = V_m \angle\theta_v$

and $\underline{I} = I_m e^{j\theta_i} = I_m \angle\theta_i$



Can see from equations, for a resistor,

$\theta_v = \theta_i$



Example (E8.5)

Current in a 4Ω resistor is known to be $\underline{I} = 12 \angle 60°$

Express the voltage across the resistor as a time function if the current freq. is 4kHz.

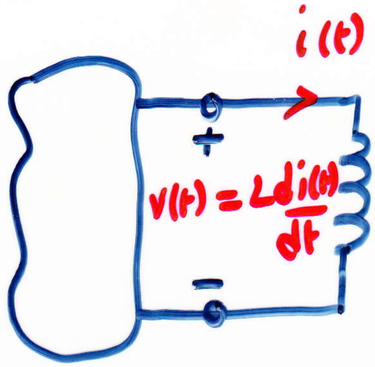
$$\underline{V} = R\,\underline{I}$$

$$\therefore \quad \underline{V} = 4 \times 12 \angle 60°$$

$$\therefore \quad v = 48 \cos(8000\pi t + 60°)$$

$$\omega = 2\pi f = 2\pi\, 4 \times 10^3$$
$$= 8000\pi$$

Inductor



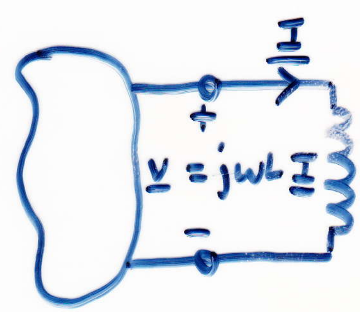
$$v(t) = L\frac{di(t)}{dt}$$

$$V_m e^{j(\omega t+\theta_v)} = L\frac{d}{dt} I_m e^{j(\omega t+\theta_i)}$$

$\rightarrow$

$$V_m e^{j\theta_v} = j\omega L I_m e^{j\theta_i}$$

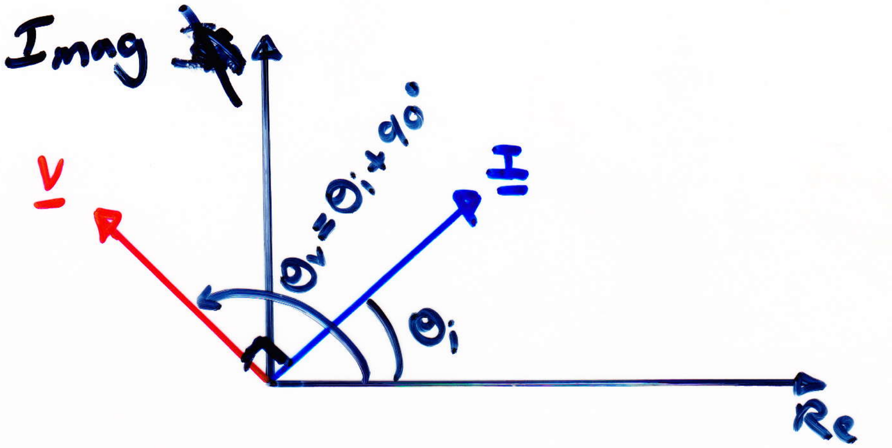
$$\underline{V} = j\omega L \underline{I}$$

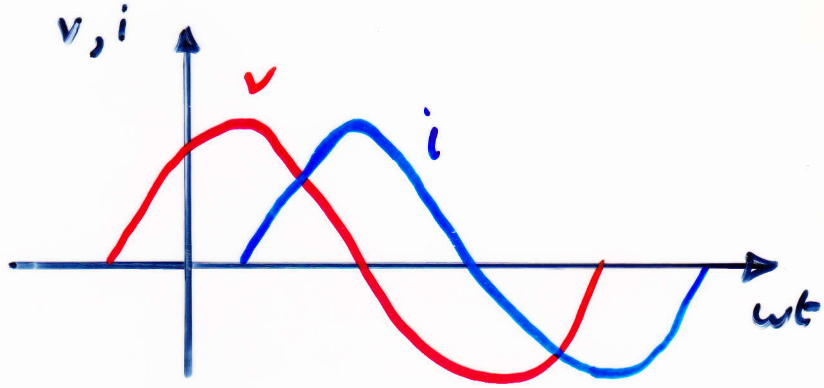


Now $j = 1e^{j90^\circ}$

$$\therefore \quad V_m e^{j\theta_v} = \omega L I_m e^{j(\theta_i+90^\circ)}$$

$$= \omega L I_m e^{j\theta_i} e^{j90}$$





Example E8.6

Current in a 0.05 H inductor is $\underline{I} = 4\angle -30^\circ$ A. If the freq. of the current is 60 Hz, determine the voltage across the inductor

$$\underline{V} = j\omega L \underline{I}$$

$$\underline{V} = j\,120\pi \times 0.05 \times 4 \angle -30^\circ$$

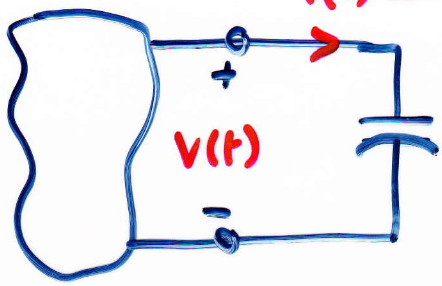
$$= j75.4 \angle -30^\circ = 75.4 \angle 90^\circ - 30^\circ$$

$$= 75.4 \angle 60^\circ$$

$$\therefore\ v_L = 75.4 \cos(120\pi t + 60^\circ)\ \text{V}$$

Capacitor

$$i(t) = C\frac{dv(t)}{dt}$$



$$i(t) = C\frac{dv(t)}{dt}$$

$$I_m e^{j(\omega t + \theta_i)} = C\frac{d}{dt}V_m e^{j(\omega t + \theta_v)}$$

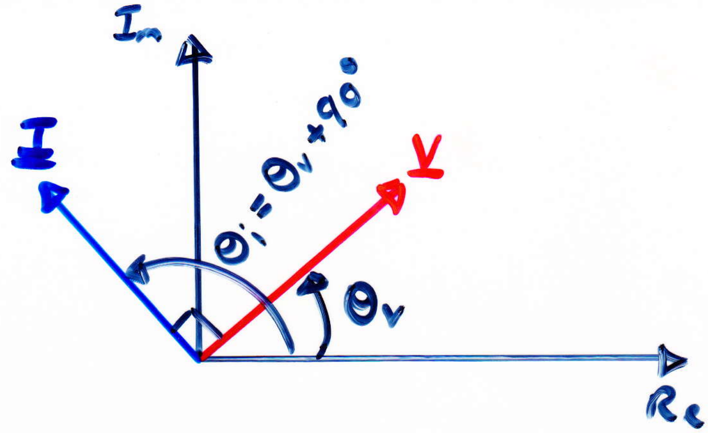
$$I_m e^{j\theta_i} = j\omega C V_m e^{j\theta_v}$$

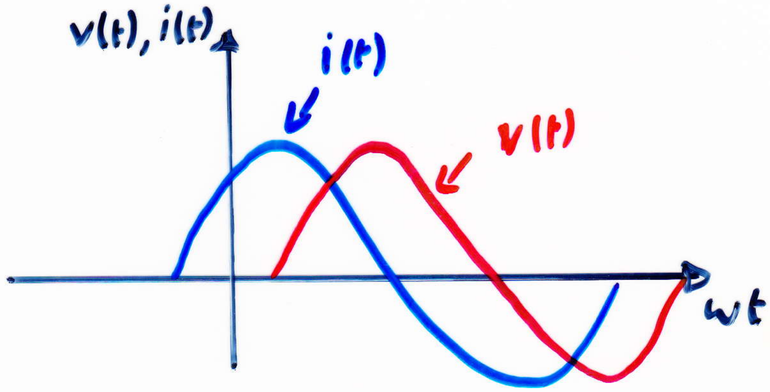
$$\underline{\underline{I}} = j\omega C \underline{V}$$

Sub. for $j = 1e^{j90^\circ}$

$$I_m e^{j\theta_i} = \omega C V_m e^{j(\theta_v + 90^\circ)}$$

voltage lags the current by 90°





Example E8.7

The current in a 150 μF capacitor is $\underline{I} = 3.6\angle{-145^\circ}$ A. If the frequency of the current is 60 Hz determine the voltage across the capacitor.

$$\underline{I} = j\omega C \underline{V}$$

$$\underline{V} = \frac{\underline{I}}{j\omega C} = \frac{3.6\angle{-145^\circ}}{\angle 90^\circ \; 120\pi \times 150 \times 10^{-6}}$$

$$= 63.66\angle{-235^\circ}$$

$$v_c = 63.66 \cos(120\pi t - 235^\circ)$$