MATH 100 (A01-A05 sections),
Midterm # 2. — October 17th, 2013
Time: 2h

Trivia: On October 17th 1983, Gerard Debreu won the Nobel Memorial Prize in Economic Sciences for applying mathematical rigor to the fundamental theory of supply and demand in economics.

Last name: Instructor

First name: Solutions

Student number: A

Section number: ____________________

Questions	Score	Out of
2 to 12		22
13		3
14		3
15		8
16		4
Total		**40**

- The only calculators we allow on any examination is the Sharp EL-510R, RN, or RNB.
- This test consists of 15 questions (numbered 2 through 16) and has 14 pages (including this cover). Questions 2 through 12 are multiple-choice. **Enter your final answer in the bubble sheet and mark them in this paper as well.** Questions 13 through 16 are long-answer. **You need to show your work for all answers, as we may disallow any answer which is not properly justified.**
- For the multiple-choice questions, select the numerical answer closest to yours. If the answer is equidistant from two nearest choices, select the largest of the two choices.
- Before starting your test enter your name, student number, and section number on this page and on the bubble sheet. **On the name field on your bubble sheet, enter your last name (use empty space for –), followed by one blank space, followed by your first name, followed by nothing else.**
- At the end of your test, turn in both this booklet and the bubble sheet.
- Enter "A" as your answer to Question 1 now.

1. **Enter "A" as your answer to Question 1 now.**

2. *[2 points]* Consider the function $f(x) = 2.5x^2 - 6x + 7.3$. Calculate $f'(2)$.

(A) -12;	(B) -3;	(C) 0;	(D) 2;	(E) 4;
(F) 6;	(G) 8;	(H) 10;	(I) 12;	(J) 14

Answer: 4 or (E)

$$f'(x) = 5x - 6$$

$$f'(2) = 5 \cdot 2 - 6 = 4$$

3. *[2 points]* Consider the function $f(x) = \dfrac{x-2}{x+2}$. Calculate $f'(-1)$.

(A) -4;	(B) -3;	(C) -2;	(D) -1;	(E) 0;
(F) 1;	(G) 2;	(H) 3;	(I) 4;	(J) 5

Answer: 4 or (I)

$$f'(x) = \frac{1 \cdot (x+2) - (x-2) \cdot 1}{(x+2)^2}$$ (quotient rule)

$$= \frac{4}{(x+2)^2}$$

$$f'(-1) = \frac{4}{(-1+2)^2} = \frac{4}{1} = 4$$

4. *[2 points]* When observations begin at $t = 0$, a cell culture has 1200 cells and continues to grow according to the function $p(t) = 1200e^t$, where p is the number of cells and t is measured in days.

On the interval $[0, 4]$, when is the growth rate of the number of cells the greatest?

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4;
(F) 5; (G) 6; (H) 10; (I) 1,200; (J) 65,500;

Answer: at time $t = 4$ or (E)

$p'(t) = 1200e^t$. From the shape of the graph of e^t (), we know that $p'(t)$ is maximized at $t = 4$, i.e., the growth rate is the greatest at $t = 4$.

5. *[2 points]* Find the slope of the tangent line to the graph of $y = \cot x$ at the point with x-coordinate $\pi/4$.

(A) -2.0; (B) -1.5; (C) -1.0; (D) -0.5; (E) 0.0;
(F) 0.5; (G) 1.0; (H) 1.5; (I) 2.0; (J) 2.5;

Answer: -2 or (A)

$y = \dfrac{\cos x}{\sin x}$, so by the quotient rule,

$$y' = \frac{-\sin x(\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$y' = \frac{-1}{\sin^2 x} \quad (\text{since } \sin^2 x + \cos^2 x = 1)$$

$$y'\left(\frac{\pi}{4}\right) = \frac{-1}{\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{-1}{1/2} = -2$$

6. *[2 points]* Calculate $\dfrac{d}{dx}\, e^{\sqrt{x}+1}$.

(A) $e^{\sqrt{x}}$; (B) $(\sqrt{x}+1)\, e^{\sqrt{x}+1}$; (C) $(\sqrt{x}+1)e^{\sqrt{x}}$; (D) $\dfrac{e^{\sqrt{x}}}{2(\sqrt{x}+1)}$; (E) $\dfrac{e^{\sqrt{x}}}{\sqrt{x}+1}$;

(F) $\sqrt{x}e^{\sqrt{x}}$; (G) $\dfrac{e^{\sqrt{x}+1}}{\sqrt{x}+1}$; (H) $\dfrac{e^{\sqrt{x}+1}}{2\,(\sqrt{x}+1)}$; (I) $\dfrac{e^{\sqrt{x}+1}}{2\sqrt{x}}$; (J) $\dfrac{e^{\sqrt{x}+1}}{\sqrt{x}}$;

Answer: (I)

Let $u(x) = \sqrt{x} + 1$, $y = e^u$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot \left(\tfrac{1}{2} x^{-1/2}\right)$$

$$= e^{\sqrt{x}+1}\left(\tfrac{1}{2} x^{-1/2}\right)$$

$$= \frac{e^{\sqrt{x}+1}}{2\sqrt{x}}$$

7. *[2 points]* What is the value of the x-coordinate of the point at which the tangent line to the graph of $f(x) = \ln(x+1)^{(x+1)}$ is horizontal?

(A) -1.00; (B) -0.65; (C) -0.35; (D) -0.15; (E) 0;
(F) 0.15; (G) 0.35; (H) 0.65; (I) 1.00;

(J) This function does not have a horizontal tangent line.

Answer: -0.6321 or (B)

Note $f(x) = (x+1)\ln(x+1)$

so by the product rule, $f'(x) = 1 \cdot \ln(x+1) + (x+1)\left(\dfrac{1}{x+1}\right)$

$$= \ln(x+1) + 1$$

We seek x such that $f'(x) = 0$, i.e.,

$$\ln(x+1) + 1 = 0$$

$$\ln(x+1) = -1$$

$$x + 1 = e^{-1}$$

$$\underline{x = e^{-1} - 1}$$

8. *[2 points]* What is the maximum value of the function $F(x) = \frac{1}{2}x - \frac{1}{8}x^3$ on the interval $[-1, 3]$?

(A) -2.0; (B) -1.5; (C) -1.0; (D) -0.5; (E) 0.0;
(F) 0.5; (G) 1.0; (H) 1.5; (I) 2.0; (J) No maximum value.

Answer: 0.3849 (when $x = 2/\sqrt{3}$) or (F)

$F'(x) = \frac{1}{2} - \frac{3}{8}x^2$

critical points of F are where $F'(x) = 0$

which is when $x^2 = \frac{4}{3}$ or when $x = \frac{\pm 2}{\sqrt{3}}$

We check the values of F there & at the endpoints:

$F(-1) = \frac{-1}{2} + \frac{1}{8} = \frac{-3}{8}$ $\quad F(-2/\sqrt{3}) = \frac{-1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} = \frac{-2}{3\sqrt{3}} \approx -0.3849$

$F(3) = \frac{3}{2} - \frac{27}{8} = \frac{-15}{8}$

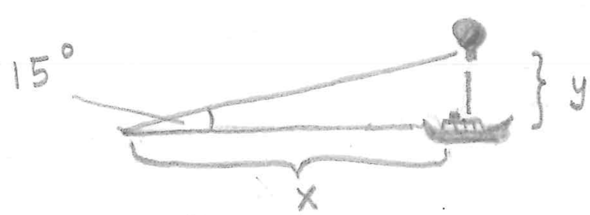
$x = -\frac{2}{\sqrt{3}}$ is not in $[-1;3]$

$F(2/\sqrt{3}) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} \approx \boxed{0.3849}$

9. *[2 points]* A surface ship is moving in a straight line at 20 km/hr. At the same time, an air balloon maintains a position directly above the ship while rising at an angle that is 15° above horizontal. How fast is the balloon's altitude increasing?

(A) 0.5 km/h; (B) 1.0 km/h; (C) 1.5 km/h; (D) 2.0 km/h; (E) 2.5 km/h;
(F) 3.0 km/h; (G) 3.5 km/h; (H) 4.0 km/h; (I) 5.0 km/h; (J) 6.0 km/h

Answer: 5.36 km/h or (I)



Given:

$y = x\tan(15°)$ $\xrightarrow{\text{take } \frac{d}{dt} \text{ of both sides}}$ $\frac{dy}{dt} = \frac{dx}{dt}\tan(15°)$

$\frac{dx}{dt} = 20$ Km/h

$= 20\tan(15°)$

$\approx 20 \cdot 0.267$

≈ 5.36 Km/h

10. *[2 points]* Find the equation of the line tangent to the graph of $x^2 + xy = y^3 + 5$ at $(2, 1)$.

(A) $y = -\frac{8}{9}x + \frac{25}{9}$; (B) $y = -\frac{1}{2}x + 2$; (C) $y = 4x - 7$; (D) $y = 5x - 9$;

(E) $y = \frac{8}{9}x - \frac{7}{9}$; (F) $y = \frac{1}{2}x$; (G) $y = -4x + 9$; (H) $y = -5x + 11$;

(I) $y = 1$; (J) None of the above;

Answer: (D)

Take $\frac{d}{dx}$ of both sides of the equation above:

$$2x + \overbrace{\left(1 \cdot y + x\frac{dy}{dx}\right)}^{\text{product rule}} = 3y^2 \frac{dy}{dx}$$

$$2x + y = (3y^2 - x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x+y}{3y^2 - x}$$

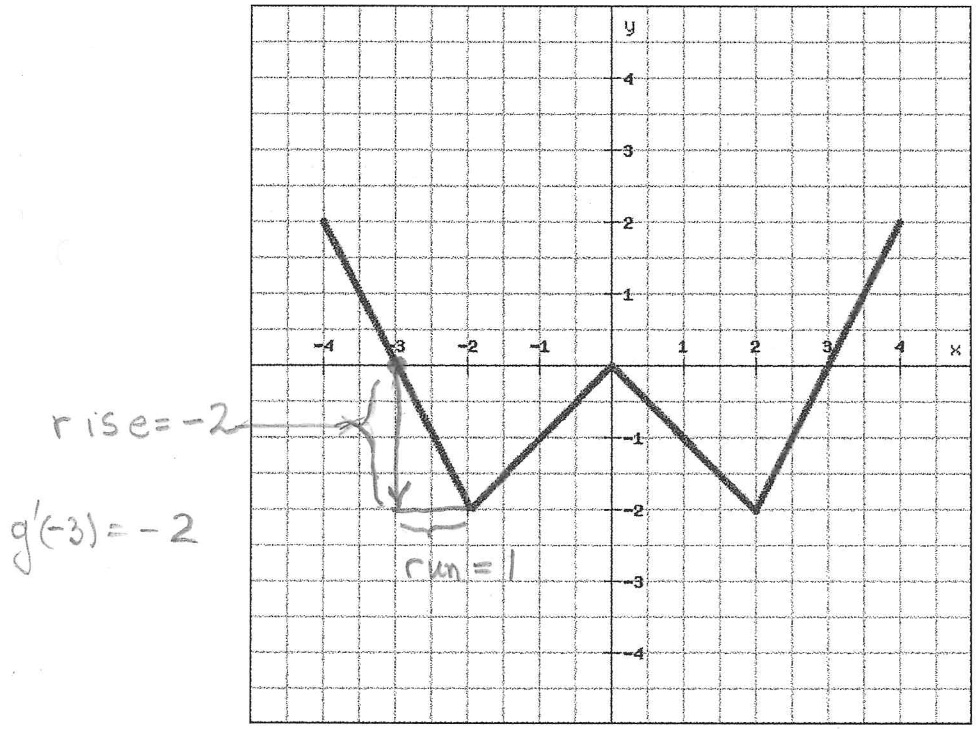
$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{2 \cdot 2 + 1}{3 \cdot 1^2 - 2} = \frac{5}{1} = 5$$

So equation of the tangent line is

$$y - 1 = 5(x - 2)$$

$$y = 5x - 9$$

11. *[2 points]* Here is the graph of a function g:



$y = g(x)$

Estimate $\dfrac{d}{dx} g\,(1 - 2x)$ when $x = 2$.

(A) −8;	(B) −4;	(C) −2;	(D) −1;	(E) 0;
(F) 1;	(G) 2;	(H) 4;	(I) 8;	(J) Undefined.

Answer: (H) or 4

Let $h(x) = g(1-2x)$, and let $f(x) = 1-2x$, so $h(x) = g(f(x))$.

By the chain rule, $h'(2) = g'(f(2))\, f'(2)$

$= g'(-3)\cdot(-2)$

(since $f(2) = 1 - 2\cdot 2$ and $f'(x) = -2$)

$= -2\cdot(-2)$

$= 4$

12. *[2 points]* The equation $x = 3\cos x$ has one of the solutions a number between 1 and 2. What is the second digit after the period of the exact solution?
(For example, if the solution were 1.9837248..., then you should choose "8" as your answer.)

Hint: One way to solve this question is to use Newton's method.

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4;
(F) 5; (G) 6; (H) 7; (I) 8; (J) 9.

Answer: (H). The correct solution is 1.17012..

We apply Newton's method to $f(x) = x - 3\cos x$, since a zero of f is a solution to the equation above. Let $x_0 = 1$ be our "first guess" (though other numbers, such as 2 or 3/2 are equally valid "first guesses").

Note $f'(x) = 1 + 3\sin x$.

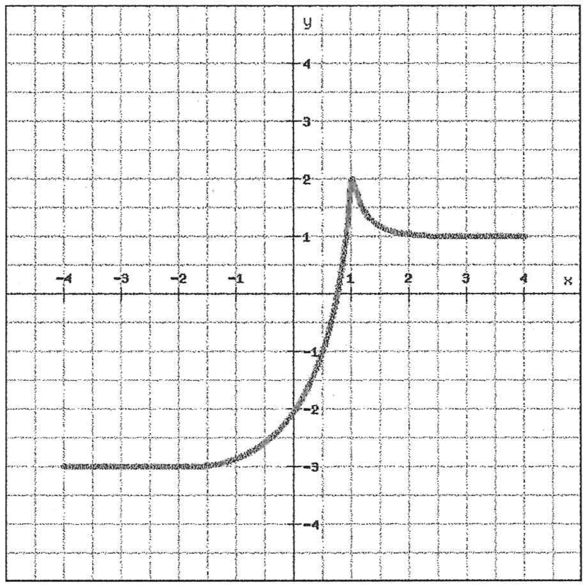
Then $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1 - 3\cos 1}{1 + 3\sin 1} \approx 1.17617...$

Then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.17617 - \frac{1.17617 - 3\cos(1.17617)}{1 + 3\sin(1.17617)} \approx 1.1701$

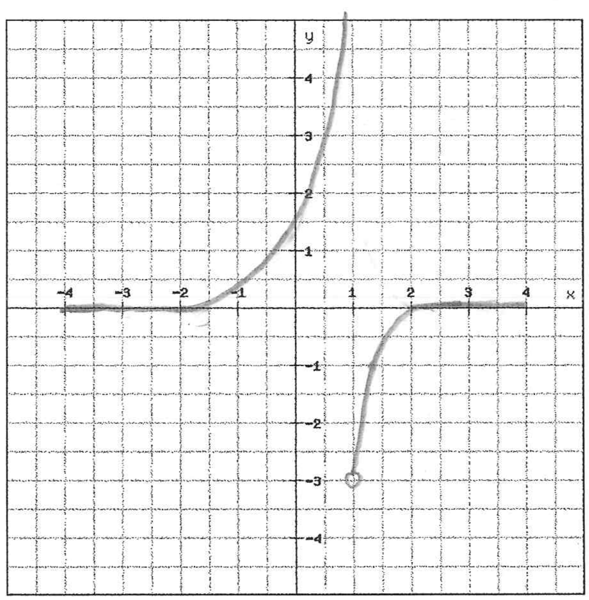
$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.1701 - \frac{1.1701 - 3\cos(1.1701)}{1 + 3\sin(1.1701)} \approx 1.1701$

At this point, the first two decimal places (and more) have "stabilized", so we can be sure that the solution is 1.17...

13. *[3 points]* Below is the graph of a function f:



Draw the graph of f':



You do not need to justify your answer for this question.

14. *[3 points]* Consider the function $g(x) = \sin x$.
Using the definition of derivative as a limit, calculate $g'(x)$.

$$g'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right)$$

$$= \lim_{h\to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= 0 \cdot \sin x + 1 \cdot \cos x$$

$$= \cos x$$

You might find one of the following formula useful:

$\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;
$\sin(2A) = 2 \sin A \cos A$

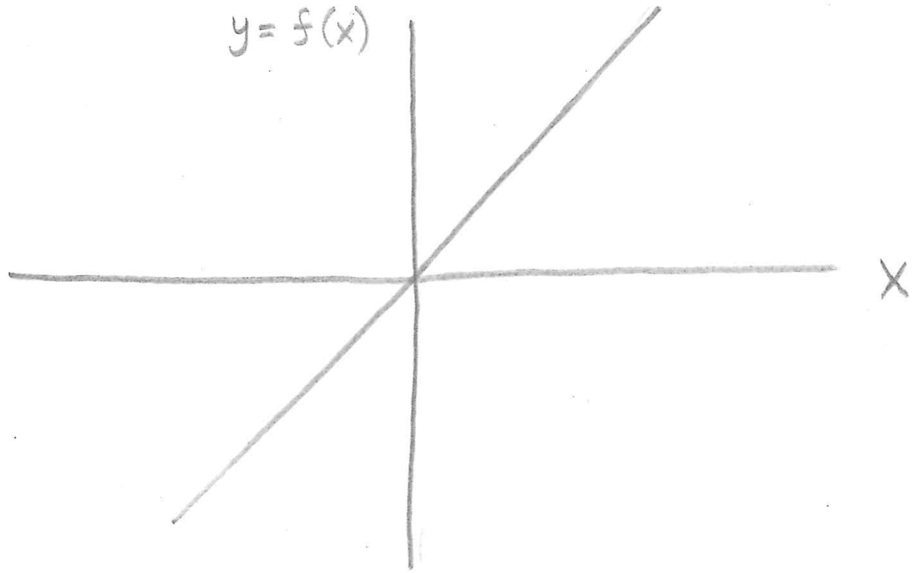
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$;
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos^2 A = 0.5 * (1 + \cos(2A))$;
$\sin^2 A = 0.5 * (1 - \cos(2A))$

15. *[8 points]*. In each of the following questions, we ask you to give us one example of a function f satisfying a certain property. There are many different correct answers to each question (including some easy ones). **Write your final answer to each question in the box and sketch its graph in the space underneath.**

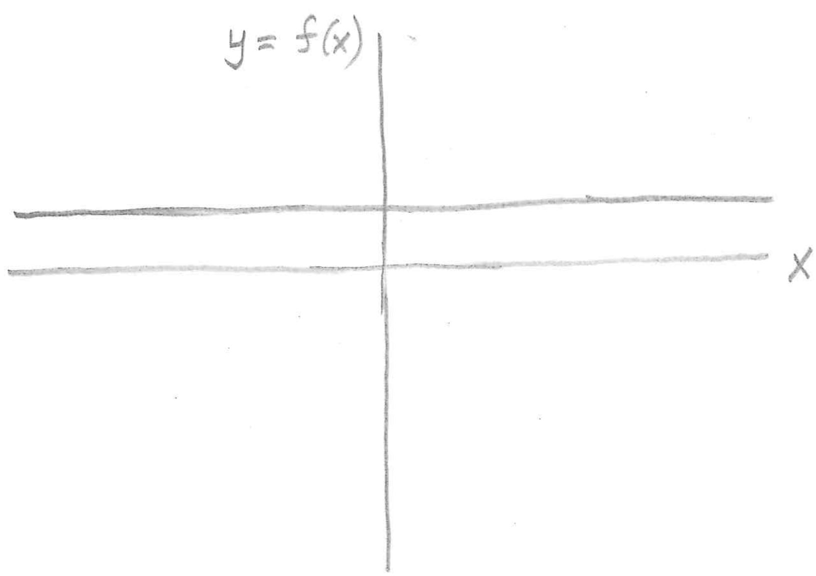
(a) Give us a function f that is continuous at every real number, such that $f'(2) = 1$.

Your answer: $f(x) =$ x



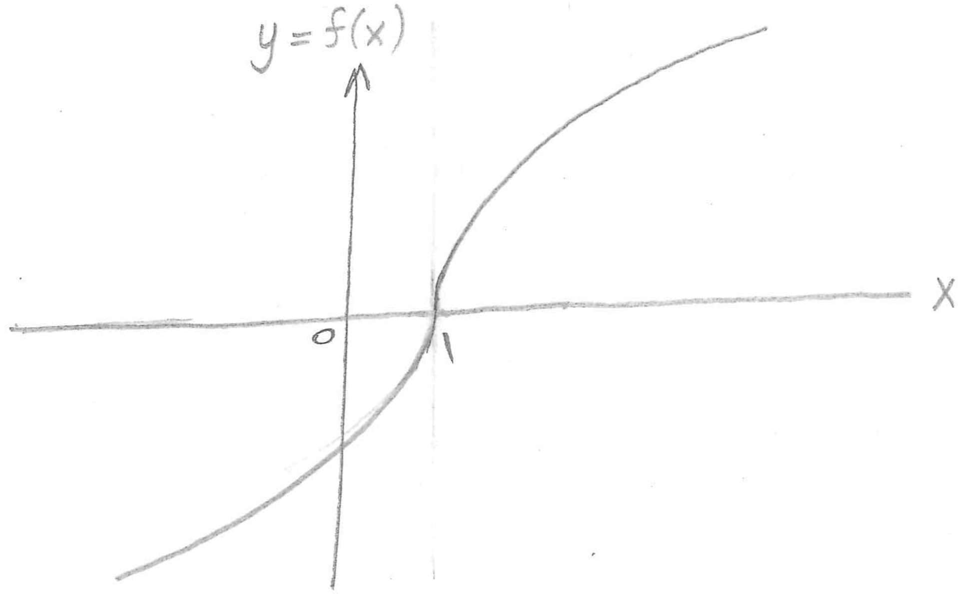
(b) Give us a function f that is defined at every real number, which has a horizontal tangent line at $x = -1$.

Your answer: $f(x) =$ 1



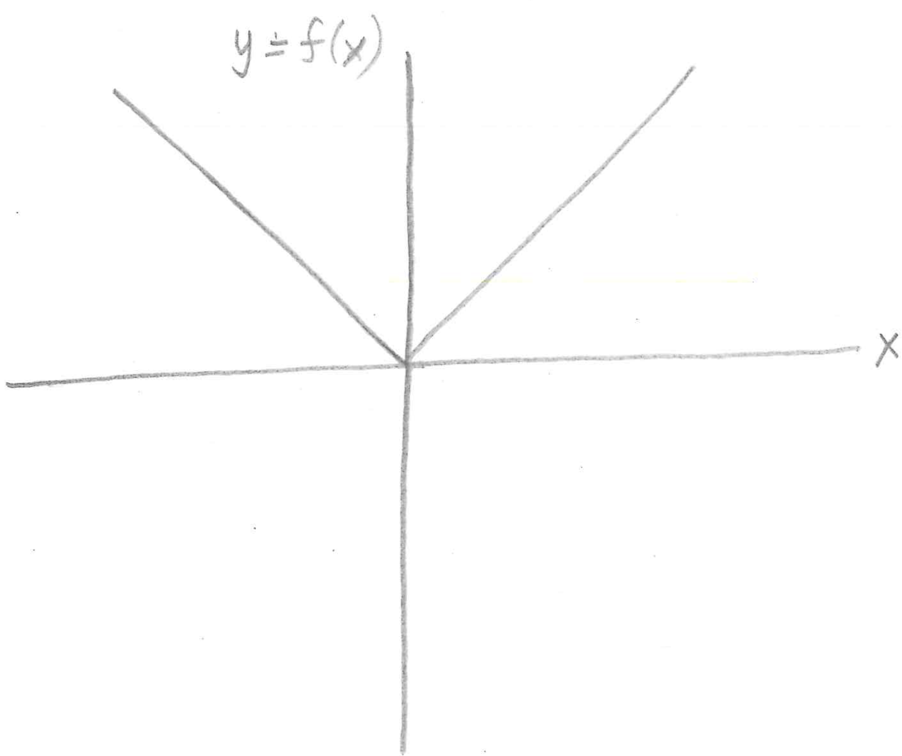
(c) Give us a function f that is defined at every real number, which has a vertical tangent line at $x = 1$.

Your answer: $f(x) = (x-1)^{1/3}$



(d) Give us a function f that is continuous at every real number, which does not have a tangent line at $x = 0$.

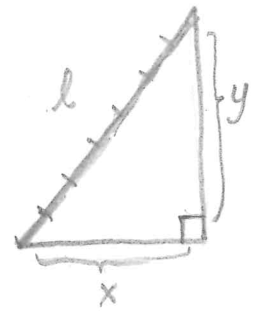
Your answer: $f(x) = |x|$



16. *[4 points]* The ladder is pushed against the wall, so that the top of a ladder slides up a vertical wall at a rate of 0.3 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides towards the wall at a rate of 0.4 m/s.

How long is the ladder?

Answer: 5 meters



Given:

constant

(*) $x^2 + y^2 = \ell^2$ (Pythagoras)

$\frac{dy}{dt} = 0.3$ m/s

$\frac{dx}{dt}(0) = -0.4$ m/s

Take $t=0$ to be the time when the bottom of the ladder is 3 m from the wall.

So $x(0) = 3$.

Take $\frac{d}{dt}$ of both sides of (*) to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

This holds for all values of t, in particular for $t=0$:

$$2 \cdot 3 \cdot (-0.4) + 2y(0) \cdot 0.3 = 0$$

$$\Rightarrow y(0) = 4.$$

Since (*) also holds at all times t,

$$x(0)^2 + y(0)^2 = \ell^2$$

$$3^2 + 4^2 = \ell^2$$

$$\Rightarrow \boxed{\ell = 5}$$

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