The University of British Columbia. Mathematics 221

Final Examination - Wednesday, December 5, 2012, 3:30-6pm

This is a closed book exam. No calculator or notes of any kind are allowed. **Each problem is worth 6 points.** In order to receive full credit for Problems 1-8 you need to show enough work to justify your answer. No explanations are required in Problems 9 and 10.

Last Name ____________________ **First** _____________ **Signature** __________________

Student Number ____________________

Please mark your section with an X

_____ Section 102. Instructor: Mathieu Huruguen

_____ Section 103. Instructor: Zinovy Reichstein

_____ Section 104. Instructor: Dale Peterson

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

(a) speaking or communicating with other candidates, unless otherwise authorized;

(b) purposely exposing written papers to the view of other candidates or imaging devices;

(c) purposely viewing the written papers of other candidates;

(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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Total	

Problem 1: Solve the following linear system

$$\begin{cases} 7x + 3y + z = 3 \\ 4x + 2y + z = 2 \\ x + y + z = 1 \end{cases}$$

Do the solutions $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of this system form a subspace of $\mathbb{R}^3$? Explain your answer.

Problem 2: Is it possible for a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ to map

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \text{ and } \quad \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} ?$$

Explain your answer.

Problem 3: Compute the determinant of the 4×4 matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Problem 4: Find a 2×2 matrix A with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$ and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Problem 5: Is the 3×3 matrix $\begin{bmatrix} -2 & 5 & -1 \\ -2 & 4 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ diagonalizable? Explain your answer.

Problem 6: Suppose $\vec{v_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ are the eigenvectors of a 3×3 matrix A, with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$, and $\lambda_3 = -10$, respectively. If the vectors $\vec{w}_0, \vec{w}_1, \vec{w}_2, \ldots$ in $\mathbb{R}^3$ are related by $\vec{w}_{n+1} = A\vec{w}_n$, and $\vec{w}_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, find a formula for $\vec{w_n}$. Your answer should be of the form $\vec{w_n} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$, where the coordinates a_n, b_n, c_n depend on n.

Hint: The eigenvectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ form an orthogonal set.

Problem 7: Let W be the plane in $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\\1\end{bmatrix}$.

(a) Find a basis for $W^\perp$.

(b) Find the vector in W closest to $\vec{y} = \begin{bmatrix}4\\0\\-5\end{bmatrix}$.

Problem 8: In the xy-plane, find the parabola $y = a + bx^2$ which best fits the data points $(0, 1)$, $(1, 1)$ and $(2, -2)$ (in the least-squares sense).

Problem 9: Multiple choice. Circle one answer in each part, no explanation required.

(1) Which assertion holds for a 3×3 matrix A satisfying $A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$?

(A) The linear system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a solution for any choice of real numbers b_1, b_2, b_3.

(B) A is not invertible.

(C) $\det(A) = 1$.

(D) One of the eigenvalues of A is 2.

(E) A^3 is the zero matrix.

Answer to part (1) (circle one): A B C D E.

(2) Which assertion holds for every pair A, B of $n \times n$ matrices?

(A) $(A + B)^2 = A^2 + B^2 + 2AB$.

(B) If A and B have the same characteristic polynomial then A and B are similar.

(C) $\det(A + B) = \det(A) + \det(B)$.

(D) If the linear system $A\vec{x} = \vec{b}$ has a solution and the linear system $B\vec{x} = \vec{b}$ has a solution then the linear system $(A + B)\vec{x} = \vec{b}$ has a solution.

(E) If the linear system $(AB)\vec{x} = \vec{b}$ has a solution then the linear system $A\vec{x} = \vec{b}$ has a solution.

Answer to part (2) (circle one): A B C D E.

(3) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation.

(A) If $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ are linearly independent vectors in $\mathbb{R}^3$ then $T(\vec{v_1})$, $T(\vec{v_2})$, $T(\vec{v_3})$ are linearly independent vectors in $\mathbb{R}^4$.

(B) $\dim \mathrm{Nul}(T) + \dim \mathrm{Range}(T) = 4$. Here $\mathrm{Nul}(T)$ denotes the null space of T.

(C) The range of T is a subspace of $\mathbb{R}^4$ of dimension ≤ 3.

(D) $T(\vec{v}) = A\vec{v}$, for some matrix A with 3 rows and 4 columns.

(E) If T is one-to-one then T is onto.

Answer to part (3) (circle one): A B C D E.

Problem 10: In the true-false questions below, A is assumed to be an $n \times n$ matrix, where $n \geq 2$. Circle one answer in each part, no explanation required.

(1) If A is invertible then the product $A^T A$ is also invertible. Here A^T denotes the transpose of A.

Answer to part (1) (circle one): True False

(2) If -1 is an eigenvalue of A then $A^2 - I_n$ is invertible. Here I_n denotes the $n \times n$ identity matrix.

Answer to part (2) (circle one): True False

(3) If A is diagonalizable then A^2 is diagonalizable.

Answer to part (3) (circle one): True False

(4) If $\det(A) = d$ then $\det(2A) = 2d$.

Answer to part (4) (circle one): True False

(5) If 3 is an eigenvalue of A then A is invertible and $\dfrac{1}{3}$ is an eigenvalue of A^{-1}.

Answer to part (5) (circle one): True False

(6) If the reduced echelon form of A has n pivots then the reduced echelon form of A^{-1} also has n pivots.

Answer to part (6) (circle one): True False