Math. 2008–2009 B, Winter 2013

TEST 2 – Solution

---

1. Find the local extrema and saddle point(s) of the function $f(x, y) = y^3 + x^3 - 3xy - 1$.

Solution. To find the critical points of f we solve $f_x = 3x^2 - 3y = 0$ and $f_y = 3y^2 - 3x = 0$, which give $x^2 = y$ and $y^2 = x$. So $y = y^4$, and thereore the solutions are $(0, 0)$ and $(1, 1)$. Next, we use the second derivative test to determine the bahaviour of the function at these critical points. For this, we calculate

$$D = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9$$

and therefore $D(0, 0) = -9 < 0$, which implies that f has a saddle point at $(0, 0)$. On the other hand $D(1, 1) = 36 - 9 > 0$, and $f_{xx}(1, 1) = 6 > 0$, which implies that $f(1, 1) = -2$ is a local minimum value for f.

2. The temperature at a point (x, y, z) of a solid ball of radius 4 with center at the origin is given by

$$T(x, y, z) = xy + yz + xz.$$

Find the direction in which T is increasing most rapidly at $(1, 1, 2)$.

Solution. The temperature is increasing most rapidly in the direction of the gradient vector ∇T at each point. We have $\nabla T(x, y, z) = \langle y + z, x + z, x + y \rangle$. So, the direction in which T is increasing most rapidly at $(1, 1, 2)$ is $\nabla T(1, 1, 2) = \langle 3, 3, 2 \rangle$ (or any parallel vector).

3. Evaluate the double integral

$$\iint_D xy\,dA$$

where D is the region enclosed by the parabola $y = x^2$ and the line $y = -2x$.

Solution. The region can be described as type I as follows:

$$D = \{\,(x,y) \,:\, -2 \le x \le 0\,,\; x^2 \le y \le -2x\,\}\,.$$

Therefore the double integral transforms to the iterated integral

$$\begin{aligned}
\iint_D xy\,dA &= \int_{-2}^{0}\int_{x^2}^{-2x} xy\;dy\,dx \\
&= \int_{-2}^{0}\left(\left.\frac{xy^2}{2}\right|_{x^2}^{-2x}\right)\,dx \\
&= \int_{-2}^{0}\left(2x^3 - \frac{x^5}{2}\right)\,dx \\
&= \left.\frac{x^4}{2} - \frac{x^6}{12}\right|_{-2}^{0} = \frac{-8}{3}\,.
\end{aligned}$$

Bonus Question

4. Evaluate the iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 \sin y^3 \, dy \, dx \, .$$

Solution. The iterated integral in its current form cannot be evaluated using elementary functions. We need to interchange the order of integration. From the limits of the iterated integral, we see that it associates to the double integral over the type I region

$$D \;=\; \{\,(x,y)\;:\;0 \le x \le 4\,,\;\sqrt{x} \le y \le 2\,\},$$

which is the region in the first quadrant enclosed by the curve $y = \sqrt{x}$ and the line $y = 2$. So, as a type II region, D is described as

$$D \;=\; \{\,(x,y)\;:\;0 \le y \le 2\,,\;0 \le x \le y^2\,\},$$

Hence, we get

$$\begin{aligned}
\int_0^4 \int_{\sqrt{x}}^2 \sin y^3 \, dy \, dx &= \int_0^2 \int_0^{y^2} \sin y^3 \, dx \, dy \\
&= \int_0^2 \left(x \sin y^3 \,\Big|_0^{y^2} \right) \, dy \\
&= \int_0^2 \left(y^2 \sin y^3 \right) \, dy \\
&= \frac{-1}{3} \cos y^3 \,\Bigg|_0^2 \\
&= \frac{1}{3}(1 - \cos 8) \, .
\end{aligned}$$