Math. 2008–2009 B, Winter 2013

TEST 1

Time: 50 minutes (Any non-programmable calculator permitted.)

Print Name : ______________________________

Student Number: ______________________________

1. **[6 marks]** Find the equation of the plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations $x = 2y = 3z$.

1 By letting $x=0$, we obtain the point $(0,0,0)$ on the line. So $(1,-1,1)-(0,0,0) = \langle 1,-1,1 \rangle$ is a vector contained in the plane. Also, from the

1 symmetric equations above, we get a parallel vector $\langle 1, \frac{1}{2}, \frac{1}{3} \rangle$ to the line. To simplify the numbers we multiply the latter vector by 6, to get $\langle 6,3,2 \rangle$ which will be as well contained in the plane. So

2 $\langle 1,-1,1 \rangle \times \langle 6,3,2 \rangle = \langle -5, 4, 9 \rangle$ is normal to the plane. Hence, an equation for the plane is: (using the point $(0,0,0)$)

2 $-5x + 4y + 9z = 0$

2. **[3 marks]** Determine which of the following equations determines the surface obtained by rotating the parabola $y = 1 + x^2$ about the y-axis.

$x^2 + y^2 + z^2 = 2$; $y = 1 + x^2$; $y = 1 + x^2 + 2z^2$

$y = 1 + x^2 + z^2$; $x^2 - 9y^2 - 9z^2 = 0$; $y = 1 + x^2 + z$

3 Using the following two facts, we can rule out other options:

1. intersection of the surface with the xy-plane (i.e. $z=0$) is the curve $y=1+x^2$.
2. intersection of the surface with any plane parallel to the xz-plane ($y=k$, $k>1$) should be a circle.

3. **[3 marks]** Find the length of the curve C given by the vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}, \qquad -2 \le t \le 1.$$

1 $\vec{r}'(t) = \langle 1, -2\sin t, 2\cos t \rangle \Rightarrow$

1 $|\vec{r}'(t)| = \sqrt{1 + 4\sin^2 t + 4\cos^2 t} = \sqrt{5} \Rightarrow$

1 $L = \int_{-2}^{1} |\vec{r}'(t)|\,dt = \int_{-2}^{1} \sqrt{5}\,dt = 3\sqrt{5}$

4. **[3 marks]** The position function of a particle is given by

$$\mathbf{r}(t) = \frac{1}{2}(t-1)^2\,\mathbf{i} + 5t\,\mathbf{j} + -7t\,\mathbf{k}.$$

When is the speed minimum?

1.5 $\vec{v}(t) = \langle t-1, 5, -7 \rangle \Rightarrow v(t) = |\vec{v}(t)| = \sqrt{(t-1)^2 + 25 + 49}$. To minimize $v(t)$, we can equivalently we can minimize $v(t)^2 = (t-1)^2 + 74$.

1.5 For this, calculate $\frac{d}{dt}\left((t-1)^2 + 74\right) = 2(t-1) = 0 \Rightarrow t = 1$.

Bonus Question

5. **[3 marks]** Find the curvature of the curve with parametric equations

$$x = \int_0^t \sin\left(\tfrac{1}{2}\pi u^2\right) du \quad ; \quad y = \int_0^t \cos\left(\tfrac{1}{2}\pi u^2\right) du.$$

1.5 By the Fundamental Theorem of Calculus we get $\vec{r}'(t) = \langle x'(t), y'(t) \rangle = \langle \sin(\frac{1}{2}\pi t^2), \cos(\frac{1}{2}\pi t^2) \rangle$

1.5 $\Rightarrow |\vec{r}'(t)| = 1 \Rightarrow k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = |\vec{r}''(t)| = |\langle \pi t\cos(\frac{1}{2}\pi t^2), -\pi t\sin(\frac{1}{2}\pi t^2) \rangle| = \pi t$.