The University of Western Ontario
CALCULUS 1500A MIDTERM EXAM 2012

Name : ______________________________

Student Number : ______________________________

INSTRUCTIONS:

1. Answer questions in the space provided.
2. Circle the correct answer for multiple choice questions.
3. In Part B, questions 17-21, of the exam you must show all work.
4. Calculators or electronic aids cannot be used.
5. Time allowed : 3 hours.

Page	Points	Score
2	6	
3	6	
4	6	
5	6	
6	6	
7	6	
8	3	
9	12	
10	10	
11	10	
12	9	
13	10	
14	10	
Total:	100	

1. (3 points) What is the supremum of the following set :

$$(0, 1] \cap [1, 5)$$

A. 1

B. 5

C. 4

D. 0

E. Does not exist

2. (3 points) Which of the following functions is NOT one-to-one

A. $f(x) = e^x$

B. $f(x) = \sqrt{x}$

C.

$$f(x) = \begin{cases} x & \textbf{\textit{if }} x \textbf{\textit{ rational}} \\ 0 & \textbf{\textit{if otherwise}} \end{cases}$$

D. $f(x) = x + 1$

E. $f(x) = \tan^{-1}(x)$

3. (3 points) $cos(tan^{-1}(x)) =$

A. $\frac{x}{\sqrt{1+x^2}}$

B. $\frac{\sqrt{1+x^2}}{x}$

C. $\frac{1}{\sqrt{1+x^2}}$

D. $\sqrt{1+x^2}$

E. x

4. (3 points) $\cos(5\pi/6) =$

A. $\sqrt{3}/2$

B. $-\sqrt{3}/2$

C. $1/2$

D. -1

E. $-1/2$

5. (3 points)

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$$

Then

A. ∞

B. $-\infty$

C. $-1/2$

D. 0

E. $1/2$

6. (3 points)

$$\lim_{x\to 2^-} \frac{|x-2|}{|x|-2} =$$

A. -1

B. 1

C. $-\infty$

D. does not exist

E. ∞

7. (3 points)

$$\lim_{x \to -\infty} (\sqrt{x^2 + 2x} + x) =$$

A. ∞

B. $-\infty$

C. 0

D. 1/2

E. -1

8. (3 points) The function below is known to be continuous :

$$f(x) = \begin{cases} ax + b & \text{if } x < 0 \\ x^2 + a + 3b & \text{if } 0 \le x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$$

Then

A. $a = -1, b = -2.$

B. $a = 2, b = -1$***.***

C. $a = b = 2.$

D. $a = 0,\ b = 1.$

E. $a = 2,\ b = 0.$

9. (3 points) If $f(x) = xe^{x^3-x^2}$ then $f'(x) =$

A.
$$xe^{x^3-x^2} + e^{x^3-x^2}$$

B.
$$e^{x^3-x^2}$$

C.
$$(1 + 3x^3 - 2x^2)e^{x^3-x^2}$$

D.
$$(1 + 3x^2 - 2x)e^{4x^2-1}$$

E.
$$(1 + 3x^2 - 2x)e^{x^3-x^2}$$

10. (3 points) If
$$y = \sin^{-1}(x^2)$$
then $y' =$

A.
$$\sin(x^2)$$

B.
$$-4x\sin^{-2}(2x)$$

C.
$$\frac{2x}{\sqrt{1-x^2}}$$

D.
$$\frac{1}{\sqrt{1-x^4}}$$

E.
$$\frac{2x}{\sqrt{1-x^4}}$$

11. (3 points)

$$\lim_{x \to \pi/4} \frac{\tan(x) - 1}{x - \pi/4} =$$

A. $f'(\pi/4)$, where $f(x) = \cos(x)$.

B. $f'(\pi/4)$, where $f(x) = \sec(x)$.

C. $f'(\pi/4)$, where $f(x) = \sec^2(x)$.

D. $f'(\pi/4)$***, where*** $f(x) = \tan(x)$***.***

E. $1/2$

12. (3 points)

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{\cos(\frac{\pi x}{2})}$$

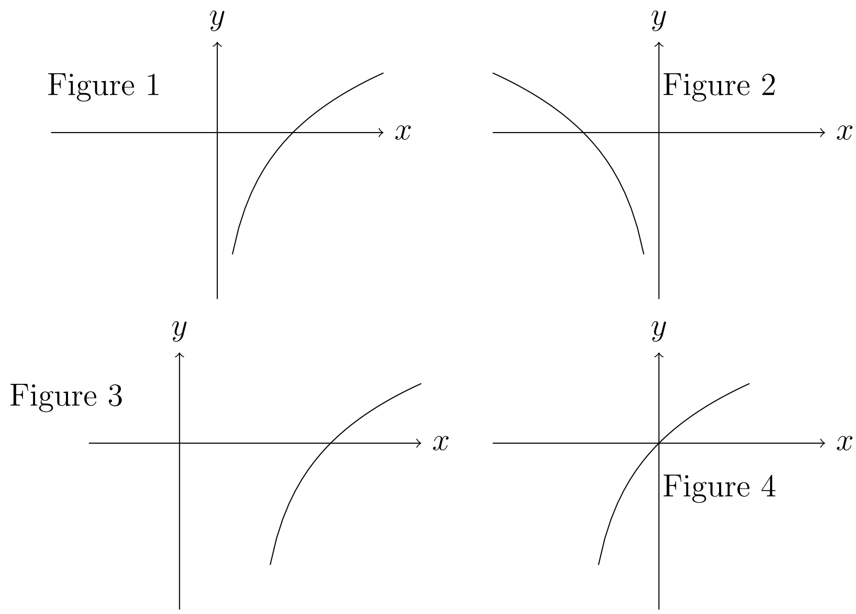
A. $-1/2$

B. 2

C. ∞

D. $-1/\pi$

E. $1/\pi$



13. (3 points) Which of the above graphs could be the graph of $y = ln(x+1)$

A. Figure 1

B. Figure 2

C. Figure 3

D. Figure 4

PART B

14. (12 points) (a) (4 points) State the $\epsilon - \delta$ definition of limit, i.e precisely define the meaning of $\lim_{x \to a} f(x) = L$.

Solution:
For every $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

(b) (4 points) State the intermediate value theorem.

Solution: Suppose that $f(x)$ is continous on $[a, b]$ and c is between $f(a)$ and $f(b)$. Then there is a $N \in (a, b)$ so that $f(N) = c$.

(c) (4 points) Suppose that S is a set of real numbers. Define the supremum of S, (i.e $\sup S$).

Solution: $\sup S$ is the least upper bound of S.

15. (10 points) Recall from lectures the following limit that was needed to compute the derivative of $\sin(x)$:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

(a) Use the above limit to show that

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0.$$

(Note : you may not use L'Hopital's rule, and you must show all work).

Solution:

$$\begin{aligned} \lim_{x\to 0} \frac{\cos(x)-1}{x} &= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\ &= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{\sin x}{\cos x + 1} \\ &= 0 \end{aligned}$$

(b) Use the above limit to calculate the limit

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} =$$

(Note : you may not use L'Hopital's rule, and you must show all work).

Solution:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} &= \frac{4}{6} \lim_{x\to 0} \frac{\sin(4x)}{4x} \frac{6x}{\sin(6x)} \\ &= \frac{4}{6} \end{aligned}$$

16. (10 points) (a) Find the equation of the tangent line to $y = xe^x$ at the point $(1, e)$.

Solution: $y' = xe^x + e^x$ so slope of tangent $= 2e$. Hence equation is

$$y - e = 2e(x - 1).$$

(b) A particle moves so that its displacement from the origin after t seconds is given by

$$s(t) = t\cos(t).$$

Find the particle's velocity after π seconds.

Solution: $v(t) = \cos(t) - t\sin(t)$. Answer $= v(\pi) = \cos(\pi) - \pi\sin(\pi) = -1$.

17. (9 points) Find y' when $x = \pi/2$ and $y = 0$ and y is defined implicitly as a function of x by the equation

$$y^2 + \cos x = e^y.$$

Solution:

$$2y'y - \sin x = y'e^y.$$

Subsitute $y = 0$ and $x = \pi/2$ to obtain $y'(0) = -\sin(\pi/2) = -1$.

There was an error in the question and the point should have been $(0, 0)$. The corrected version gives $y' = 0$. Either answer will do.

18. (10 points) Consider the function

$$f(x) = \begin{cases} x^3 & \text{if } x \text{ is rational} \\ 0 & \text{if otherwise} \end{cases} .$$

Show that $f(x)$ is differentiable at $x = 0$ and calculate its derivative $f'(0)$.

Solution: We have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h)}{h} \end{aligned}$$

Now

$$0 \leq f(h)/h \leq h^2.$$

As

$$0 = \lim_{h \to 0} h^2 = \lim_{h \to 0} 0$$

we may apply the squeeze theorem to conclude that

$$0 = \lim_{h \to 0} f(h)/h = f'(0).$$

As the limit exists the function is differentiable, further we have calculated its derivative to be 0.

A common error was to write something like :

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h^3}{h} \end{aligned}$$

which is wrong, and would lose some marks. The number of marks lost depends upon the severity of the error.

19. (10 points) Show that

$$\cos x \geq 1 - x$$

when $x > 0$.

Solution: We apply the the mean value theorem to the differentiable function $f(x) = cosx$ on the interval $[0, x]$ with $x > 0$. So there is a $c \in (0, x)$ so that

$$\begin{aligned} \frac{\cos(x) - \cos(0)}{x - 0} &= -\sin(c) \\ \text{as } -\sin(c) \geq -1 & \\ \frac{\cos(x) - \cos(0)}{x - 0} &\geq -1 \end{aligned}$$

Rearranging this we obtain $\cos(x) \geq -x + 1$.