Tutorial 5 Solutions $1104D$, Fall 2011

1. Mark each statement True or False:

$a:$ T $F:$ The columns of any 5×6 matrix are linearly dependent.

$b:$ T $F:$ The columns of a matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.

Solution:
The first part is true. The second part is wrong.

2. Find the solution set of the following linear system by Cramer's rule.

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 - 3x_2 - x_3 = 0 \\ -x_1 - 2x_2 + x_3 = -3 \end{cases}$$

Solution:
First of all we have to fine $\det(A)$, and $\det(A_i)$ for $1 \leq i \leq 3$.

$$\det(A) = \det \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & -1 \\ -1 & -2 & 1 \end{pmatrix} = -11$$

$$\det(A_1) = \det \begin{pmatrix} 2 & -1 & 1 \\ 0 & -3 & -1 \\ -3 & -2 & 1 \end{pmatrix} = -22$$

$$\det(A_2) = \det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -1 & -3 & 1 \end{pmatrix} = -11$$

$$\det(A_3) = \det \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 0 \\ -1 & -2 & -3 \end{pmatrix} = -11.$$

It means that

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-22}{-11} = 2,$$

and

$$x_2 = x_3 = \frac{\det(A_2)}{\det(A)} = \frac{\det(A_3)}{\det(A)} = \frac{-11}{-11} = 1.$$

3. Find $(A^{-1})_{2,3} + (A^{-1})_{1,2}$ where

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Only solutions which use $\text{adj}(A)$ are acceptable.
Solution:

$$\det(A) = \det \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} = 8.$$

Based on the definition of $\text{adj}(A)$, we get

$$\begin{aligned} \left(A^{-1}\right)_{2,3} + \left(A^{-1}\right)_{1,2} &= \frac{1}{\det(A)}(-1)^{3+2}\det(A_{3,2}) + \frac{1}{\det(A)}(-1)^{2+1}\det(A_{2,1}) \\ &= \frac{1}{8}(-1)\det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \frac{1}{8}(-1)\det\begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix} \\ &= \frac{-1}{8}((1)(1)-(0)(2)) + \frac{-1}{8}((-2)(3)-(0)(1)) \\ &= \frac{-1}{8}(1) + \frac{-1}{8}(-6) \\ &= \frac{5}{8}. \end{aligned}$$

4. Let the vectors $\vec{v}_1, \ldots, \vec{v}_4$ are given as follows:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Do these vectors form a basis for $\mathbb{R}^4$? Explain it.
(b) Are these vectors linearly independent in $\mathbb{R}^4$? Explain your answer.

Solution:

$$\begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 1 \\ 2 & 0 & -2 & 1 \\ 0 & 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Since every column of REF form of A has a pivot position, these vectors are linearly dependent in $\mathbb{R}^4$