

**MAT 2322 A
CALCULUS III
MIDTERM # 2
November 22, 2012**

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: _____

Solutions

Student Number: _____

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 5 pages.
- There are 4 questions worth a total of 20 marks.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

A

Question 1. (4 points) A moving object has position vector

$$\vec{r}(t) = (9t^2 - 18t + 5)\hat{i} + (2t^3 - 6t)\hat{j} + (3t^2 - 6t - 2)\hat{k}.$$

(a) Find the velocity and acceleration vectors.

(b) Are there any times at which the object is at rest? If yes, what is the acceleration at those times?

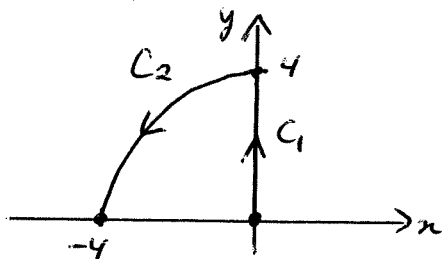
a)
$$\vec{v}(t) = \vec{r}'(t) = (18t - 18)\hat{i} + (6t^2 - 6)\hat{j} + (6t - 6)\hat{k}$$

$$\vec{a}(t) = \vec{v}'(t) = 18\hat{i} + 12t\hat{j} + 6\hat{k}$$

b)
$$\vec{v}(t) = \vec{0} \quad \text{if } t = 1$$

$$\vec{a}(1) = 18\hat{i} + 12\hat{j} + 6\hat{k}$$

Question 2. (5 points) Calculate the line integral of $\vec{F}(x, y) = 2x\hat{i} + (1 - 2y)\hat{j}$ along the path C , where C consists of the straight line from $(0, 0)$ to $(0, 4)$ and then a circular arc (part of the circle $x^2 + y^2 = 16$) directly from $(0, 4)$ to $(-4, 0)$.



C_1 is $x=0, y=t \quad 0 \leq t \leq 4$
 $\vec{r}(t) = t\hat{j}, \quad \vec{r}'(t) = \hat{j}$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^4 (1-2t)\hat{j} \cdot \hat{j} dt$$

$$= \int_0^4 (1-2t) dt = t - t^2 \Big|_0^4 = \boxed{-12}$$

C_2 is $x = -4\sin t, y = 4\cos t \quad 0 \leq t \leq \pi/2$
 $\vec{r}(t) = -4\sin t \hat{i} + 4\cos t \hat{j}, \quad \vec{r}'(t) = -4\cos t \hat{i} - 4\sin t \hat{j}$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} (-4\sin t \hat{i} + (1-8\cos t)\hat{j}) \cdot (-4\cos t \hat{i} - 4\sin t \hat{j}) dt$$

$$= \int_0^{\pi/2} (16\sin t \cos t - 4\sin t) dt$$

$$= 32 \sin^2 t + 4\cos t \Big|_0^{\pi/2} = (32+0) - (0+4) = \boxed{28}$$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 28 - 12 = \boxed{16}$

OR
 easier way: since $\vec{F} = 2x\hat{i} + (1-2y)\hat{j} = \nabla f$ where
 $f(x,y) = x^2 + y - y^2$, \vec{F} is path-independent and
 we can choose any path from $(0,0)$ to $(-4,0)$, like the
 straight line $\vec{r}(t) = -t\hat{i}, \quad 0 \leq t \leq 4, \quad \vec{r}'(t) = -\hat{i}$
 so $\int_C \vec{F} \cdot d\vec{r} = \int_0^4 (-2t\hat{i}) \cdot (-\hat{i}) dt = \int_0^4 2t dt = t^2 \Big|_0^4 = \boxed{16}$

OR easiest way: since $\vec{F} = \nabla f$, where $f(x,y) = x^2 + y - y^2$
 $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(-4,0) - f(0,0)$
 $= (-4)^2 + 0 - 0 = \boxed{16}$

Question 3. (6 points) Exactly one of the following vector fields is a gradient field. Determine which one it is and explain why the others are not gradient vector fields. What is the potential function for the gradient vector field?

$$\vec{F}(x, y) = \sin(y)\hat{i} + \cos(y)\hat{j}$$

$$\vec{G}(x, y) = xy^2\hat{i} - x^2\hat{j}$$

$$\vec{H}(x, y) = (2x + \sin(y))\hat{i} + x\cos(y)\hat{j}$$

all of the vector fields are defined in all of \mathbb{R}^2 ,
so we can check if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ or not

$$\vec{F}(x, y) = \sin y \hat{i} + \cos y \hat{j} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - \cos y \neq 0$$

not gradient field

$$\vec{G} = xy^2 \hat{i} - x^2 \hat{j} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2xy \neq 0$$

not gradient field

$$\vec{H} = (2x + \sin y)\hat{i} + x\cos y \hat{j} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos y - \cos y = 0$$

so \vec{H} is gradient field

$$\vec{H} = \nabla h \quad \text{so } h(x, y) = \int (2x + \sin y) dx + g(y)$$

$$= x^2 + x \sin y + g(y)$$

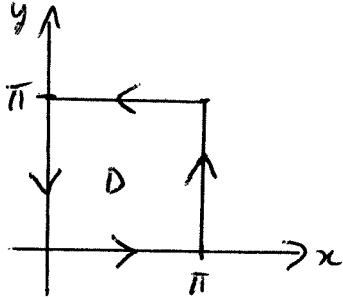
then $\frac{dh}{dy} = x \cos y + g'(y) = Q = x \cos y \Rightarrow g'(y) = 0$

so $g(y) = C$

\therefore $h(x, y) = x^2 + x \sin y + C$ is the potential function

(A)

Question 4. (5 points) Let C be the square with corners at $(0,0)$, $(\pi,0)$, (π,π) , and $(0,\pi)$, oriented counter-clockwise. If $\vec{F}(x,y) = x \cos(y) \hat{i} + y \sin(x) \hat{j}$, calculate $\oint_C \vec{F} \cdot d\vec{r}$ using Green's Theorem.



Green's Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$$P(x,y) = x \cos y$$

$$Q(x,y) = y \sin x$$

so
$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_0^\pi \int_0^\pi (y \cos x + x \sin y) dx dy \\ &= \int_0^\pi \left(y \sin x + \frac{1}{2} x^2 \sin y \Big|_0^\pi \right) dy \\ &= \int_0^\pi \frac{1}{2} \pi^2 \sin y dy \\ &= \frac{1}{2} \pi^2 \left(-\cos y \Big|_0^\pi \right) \\ &= \frac{1}{2} \pi^2 (2) \\ &= \boxed{\pi^2} \end{aligned}$$

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Question 1. (4 points) A moving object has position vector

$$\vec{r}(t) = (2t^3 - 6t - 3)\hat{i} + (3t^2 - 6t + 4)\hat{j} + (9t^2 - 18t + 2)\hat{k}.$$

(a) Find the velocity and acceleration vectors.

(b) Are there any times at which the object is at rest? If yes, what is the acceleration at those times?

a)
$$\vec{v}(t) = \vec{r}'(t) = (6t^2 - 6)\hat{i} + (6t - 6)\hat{j} + (18t - 18)\hat{k}$$

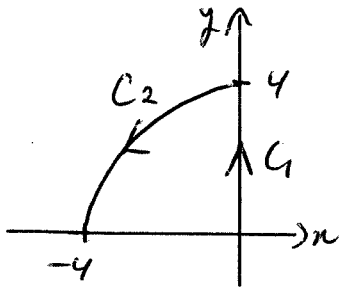
$$\vec{a}(t) = \vec{v}'(t) = 12t\hat{i} + 6\hat{j} + 18\hat{k}$$

b)
$$\vec{v}(t) = \vec{0} \quad \text{if } t = 1$$

$$\vec{a}(1) = 12\hat{i} + 6\hat{j} + 18\hat{k}$$

(3)

Question 2. (5 points) Calculate the line integral of $\vec{F}(x, y) = 4x\hat{i} + (2 - 2y)\hat{j}$ along the path C , where C consists of the straight line from $(0, 0)$ to $(0, 4)$ and then a circular arc (part of the circle $x^2 + y^2 = 16$) directly from $(0, 4)$ to $(-4, 0)$.



$$C_1 \quad x=0, \quad y=t \quad 0 \leq t \leq 4$$

$$\vec{r}(t) = t\hat{j}, \quad \vec{r}'(t) = \hat{j}$$

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^4 (2 - 2t)\hat{j} \cdot \hat{j} dt = \int_0^4 (2 - 2t) dt \\ &= 2t - t^2 \Big|_0^4 = \boxed{-8} \end{aligned}$$

$$C_2 \quad x = -4\sin t, \quad y = 4\cos t, \quad 0 \leq t \leq \pi/2$$

$$\vec{r}(t) = -4\sin t \hat{i} + 4\cos t \hat{j}, \quad \vec{r}'(t) = -4\cos t \hat{i} - 4\sin t \hat{j}$$

$$\begin{aligned} \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^{\pi/2} (-16\sin t \hat{i} + (2 - 8\cos t)\hat{j}) \cdot (-4\cos t \hat{i} - 4\sin t \hat{j}) dt \\ &= \int_0^{\pi/2} (96\sin t \cos t - 8\sin t) dt \end{aligned}$$

$$= 48\sin^2 t + 8\cos t \Big|_0^{\pi/2} = (48+0) - (0+8) = \boxed{40}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 40 - 8 = \boxed{32}$$

OR easier way: $\vec{F} = 4x\hat{i} + (2 - 2y)\hat{j} = \nabla f$ where

$f(x, y) = 2x^2 + 2y - y^2$, so \vec{F} is path-independent and we can take any path from $(0, 0)$ to $(-4, 0)$, like the straight line $\vec{r}(t) = -t\hat{i}$ $0 \leq t \leq 4$, $\vec{r}'(t) = -\hat{i}$

$$\text{so } \int_C \vec{F} \cdot d\vec{r} = \int_0^4 (-4t\hat{i}) \cdot (-\hat{i}) dt = \int_0^4 4t dt = 2t^2 \Big|_0^4 = \boxed{32}$$

OR easiest way: $\vec{F} = \nabla f$ where $f(x, y) = 2x^2 + 2y - y^2$

$$\text{then } \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(-4, 0) - f(0, 0)$$

$$= 2(-4)^2 + 0 - 0 = \boxed{32}$$

Question 3. (6 points) Exactly one of the following vector fields is a gradient field. Determine which one it is and explain why the others are not gradient vector fields. What is the potential function for the gradient vector field?

$$\vec{F}(x, y) = xy^3 \hat{i} + x^2 \hat{j}$$

$$\vec{G}(x, y) = (2x - \cos(y)) \hat{i} + x \sin(y) \hat{j}$$

$$\vec{H}(x, y) = \cos(y) \hat{i} + \sin(x) \hat{j}$$

The vector fields are defined everywhere in \mathbb{R}^2 , so we can check if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ or not

$$\vec{F} = xy^3 \hat{i} + x^2 \hat{j} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 3xy^2 \neq 0$$

not gradient field

$$\vec{G} = (2x - \cos y) \hat{i} + x \sin y \hat{j} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin y - \sin y = 0$$

so \vec{G} is gradient field

$$\vec{H} = \cos y \hat{i} + \sin x \hat{j} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos x + \sin y \neq 0$$

not gradient field

$$\vec{G} = \nabla g \quad \text{so} \quad g(x, y) = \int (2x - \cos y) dx + f(y)$$

$$= x^2 - x \cos y + f(y)$$

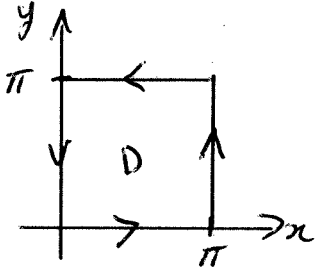
$$\text{then } \frac{\partial g}{\partial y} = x \sin y + f'(y) = Q = x \sin y \Rightarrow f'(y) = 0$$

$$\text{thus } f(y) = C$$

$$\therefore \boxed{g(x, y) = x^2 - x \cos y + C} \quad \text{is the potential function}$$

(B)

Question 4. (5 points) Let C be the square with corners at $(0,0)$, $(\pi,0)$, (π,π) , and $(0,\pi)$, oriented counter-clockwise. If $\vec{F}(x,y) = x \cos(y) \hat{i} + 2y \sin(x) \hat{j}$, calculate $\oint_C \vec{F} \cdot d\vec{r}$ using Green's Theorem.



Green's Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$$P(x,y) = x \cos y$$

$$Q(x,y) = 2y \sin x$$

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^\pi \int_0^\pi (2y \cos x + x \sin y) dx dy \\ &= \int_0^\pi \left(2y \sin x + \frac{1}{2} x^2 \sin y \Big|_0^\pi \right) dy \\ &= \int_0^\pi \frac{1}{2} \pi^2 \sin y dy \\ &= \frac{1}{2} \pi^2 (-\cos y \Big|_0^\pi) \\ &= \boxed{\pi^2} \end{aligned}$$