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MAT 2322 A
CALCULUS III
MIDTERM
October 11, 2012

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:____________________

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 6 pages.
- There are 5 questions worth 4 marks each for a total of 20 marks.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

A

Question 1. (4 points) Find and classify the critical points of the function $f(x, y) = x^3 + y^2 - 2xy$.

$f_x = 3x^2 - 2y$

$f_y = 2y - 2x = 2(y-x)$

so $f_y = 0$ only if $x = y$

so then $f_x = 3x^2 - 2x = x(3x-2) = 0$ if $x = 0, 2/3$

thus there are 2 critical pts $(0,0)$ and $(2/3, 2/3)$

$f_{xx} = 6x, \quad f_{xy} = -2, \quad f_{yy} = 2$

so $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 12x - 4$

$D(0,0) < 0 \quad \Rightarrow \quad \boxed{(0,0) \text{ is a saddle pt}}$

$D(2/3, 2/3) > 0, \quad f_{xx}(2/3, 2/3) > 0$

$\Rightarrow \quad \boxed{(2/3, 2/3) \text{ is a local min}}$

(A)

Question 2. (4 points) Use Lagrange Multipliers to find the absolute maximum and minimum values of the function $f(x,y) = (x+y)^2$ subject to the constraint $x^2+y^2 = 2$.

solve $\nabla f = \lambda \nabla g$ or $f_x = \lambda g_x$

$g = c$ $f_y = \lambda g_y$

$g(x,y) = x^2+y^2 = 2$

$f_x = \lambda g_x \Rightarrow 2(x+y) = 2\lambda x \Rightarrow x+y = \lambda x$

$f_y = \lambda g_y = 2(x+y) = 2\lambda y \Rightarrow x+y = \lambda y$

so either $x=y$ or $\lambda = 0$ and $x=-y$

$x^2+y^2 = 2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

and there are 4 pts $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$

$f(1,1) = f(-1,-1) = 4$ max

$f(-1,1) = f(1,-1) = 0$ min

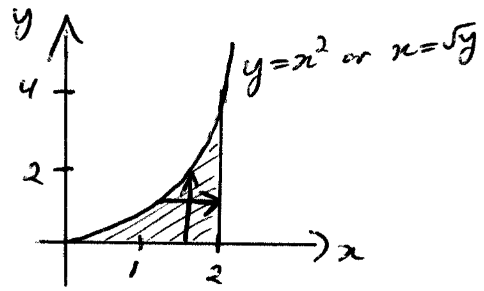
so $\boxed{\text{max is } 4}$ and $\boxed{\text{min is } 0}$

A

Question 3. (4 points) Evaluate the following integral

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{2}{1+x^3}\, dx\, dy\,.$$

have to switch the order of integration



x goes $\sqrt{y}$ to 2

then y goes 0 to 4

so then y goes from 0 to x^2

and then x is 0 to 2

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{2}{1+x^3}\, dx\, dy = \int_0^2 \int_0^{x^2} \frac{2}{1+x^3}\, dy\, dx$$

$$= \int_0^2 \frac{2}{1+x^3} \left(y \Big|_0^{x^2} \right) dx$$

$$= \int_0^2 \frac{2x^2}{1+x^3}\, dx$$

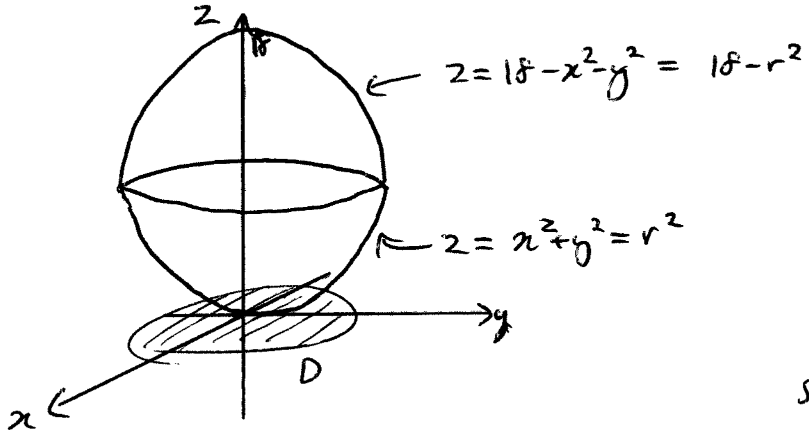
$$= \frac{2}{3} \ln (1+x^3) \Big|_0^2$$

$$= \boxed{\frac{2}{3} \ln 9}$$

$$\simeq \boxed{1.4648}$$

(A)

Question 4. (4 points) Find the volume of the solid enclosed by the paraboloids $z = 18 - x^2 - y^2$ and $z = x^2 + y^2$.



intersection

$$18 - x^2 - y^2 = x^2 + y^2$$

$$x^2 + y^2 = 9$$

so D is disk of radius 3

use polar coordinates

Volume $\quad V = \int_0^{2\pi} \int_0^3 (18 - r^2 - r^2)\, r\,dr\,d\theta$

$$= 2\pi \int_0^3 (18 - 2r^2) r\,dr$$

$$= 2\pi \int_0^3 (18r - 2r^3)\,dr$$

$$= 2\pi \left(9r^2 - \frac{1}{2} r^4 \Big|_0^3 \right)$$

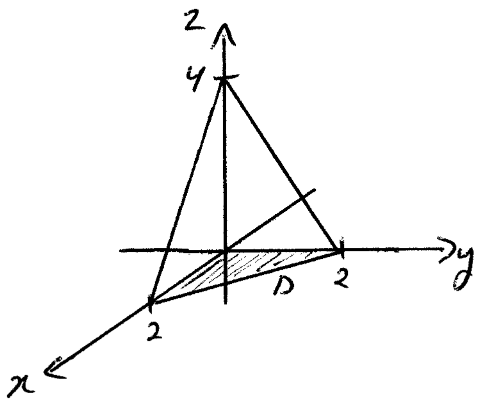
$$= 2\pi \left(81 - \frac{81}{2} \right)$$

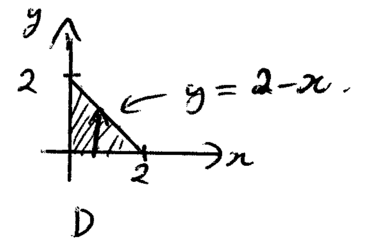
$$= \boxed{81\pi}$$

$$\simeq \boxed{254.5}$$

A

Question 5. (4 points) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + 2y + z = 4$.





volume is $V = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dz\, dy\, dx$

$$= \int_0^2 \int_0^{2-x} (4-2x-2y)\, dy\, dx$$

$$= \int_0^2 \left((4-2x)y - y^2 \Big|_0^{2-x} \right) dx$$

$$= \int_0^2 \left(2(2-x)^2 - (2-x)^2 \right) dx$$

$$= \int_0^2 (2-x)^2\, dx$$

$$= \frac{-1}{3}(2-x)^3 \Big|_0^2$$

$$= \boxed{8/3}$$

(B)

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(B)

Question 1. (4 points) Find and classify the critical points of the function $f(x, y) = 2x^3 + y^2 - 2xy$.

$f_x = 6x^2 - 2y$

$f_y = 2y - 2x = 2(y - x)$

$f_y = 0$ only if $x = y$

so then $f_x = 6x^2 - 2x = 2x(3x - 1) = 0$ if $x = 0, 1/3$

thus there are 2 critical pts $(0,0)$, $(1/3, 1/3)$

$f_{xx} = 12x$, $f_{xy} = -2$, $f_{yy} = 2$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 24x - 4$

$D(0,0) < 0 \Rightarrow$ $\boxed{(0,0) \text{ saddle pt}}$

$D(1/3, 1/3) > 0$, $f_{xx}(1/3, 1/3) > 0$

$\Rightarrow$ $\boxed{(1/3, 1/3) \text{ local min}}$

(B)

Question 2. (4 points) Use Lagrange Multipliers to find the absolute maximum and minimum values of the function $f(x,y) = (x+y)^2$ subject to the constraint $x^2+y^2 = 8$.

$f_x = \lambda g_x \Rightarrow 2(x+y) = 2\lambda x \Rightarrow x+y = \lambda x$

$f_y = \lambda g_y \Rightarrow 2(x+y) = 2\lambda y \Rightarrow x+y = \lambda y$

either $x=y$ or $\lambda=0$ and $x=-y$

$x^2+y^2=8 \Rightarrow 2x^2=8 \Rightarrow x=\pm 2$

4 points $(2,2)$, $(2,-2)$, $(-2,2)$ and $(-2,-2)$

$f(2,2) = f(-2,-2) = 16$ max

$f(2,-2) = f(-2,2) = 0$ min

$\therefore$ $\boxed{\text{max is } 16 \quad \text{min is } 0}$

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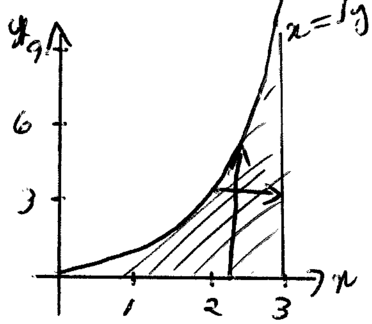
Question 3. (4 points) Evaluate the following integral

$$\int_0^9 \int_{\sqrt{y}}^3 \frac{1}{1+x^3}\, dx\, dy\,.$$

switch order



x goes $\sqrt{y}$ to 3

then y is 0 to 9

so y goes from 0 to x^2

and then x is 0 to 3

so $\displaystyle\int_0^9 \int_{\sqrt{y}}^3 \frac{1}{1+x^3}\, dx\,dy = \int_0^3 \int_0^{x^2} \frac{1}{1+x^3}\, dy\, dx$

$$= \int_0^3 \frac{x^2}{1+x^3}\, dx$$

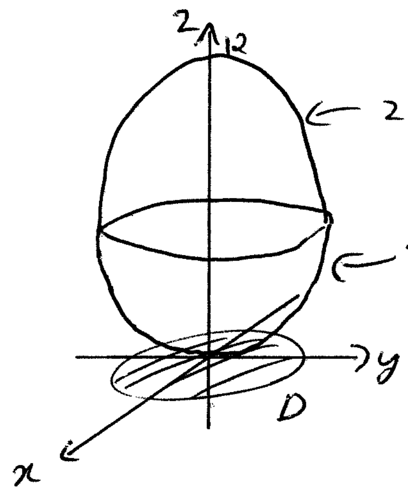
$$= \frac{1}{3} \ln (1+x^3) \Big|_0^3$$

$$= \boxed{\frac{1}{3} \ln 28}$$

$$\simeq \boxed{1.1107}$$

(B)

Question 4. (4 points) Find the volume of the solid enclosed by the paraboloids $z = 12 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.



$\leftarrow z = 12 - x^2 - y^2 = 12 - r^2$

$\leftarrow z = 2x^2 + 2y^2 = 2r^2$

intersection

$12 - x^2 - y^2 = 2x^2 + 2y^2$

$x^2 + y^2 = 4$

D is disk of radius 2

use polar coordinates

$$V = \int_0^{2\pi} \int_0^2 (12 - r^2 - 2r^2) r \, dr \, d\theta$$

$$= 2\pi \int_0^2 (12r - 3r^3) \, dr$$

$$= 2\pi \left(6r^2 - \frac{3}{4} r^4 \Big|_0^2 \right)$$

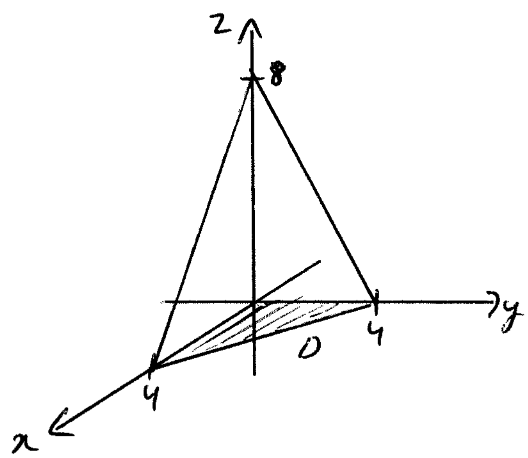
$$= 2\pi (24 - 12)$$

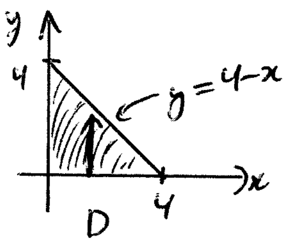
$$= \boxed{24\pi}$$

$$\simeq \boxed{75.4}$$

Ⓑ

Question 5. (4 points) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + 2y + z = 8$.





$$V = \int_0^4 \int_0^{4-x} \int_0^{8-2x-2y} dz\, dy\, dx$$

$$= \int_0^4 \int_0^{4-x} (8-2x-2y)\, dy\, dx$$

$$= \int_0^4 \left((8-2x)y - y^2 \Big|_0^{4-x} \right) dx$$

$$= \int_0^4 \left(2(4-x)^2 - (4-x)^2 \right) dx$$

$$= \int_0^4 (4-x)^2\, dx$$

$$= -\frac{1}{3}(4-x)^3 \Big|_0^4$$

$$= \boxed{\frac{64}{3}}$$