Name: **ID:**

McGill University

Probability Theory (MATH323A)

Mid-Term Exam, Tuesday October 23, 2012

NOTE: *This test consists of four questions. There is no optional question.*

Question 1. Suppose that we ask n randomly selected people whether they share your birthday.

- (a) Give an expression for the probability that no one shares your birthday (ignore leap years).(*3 marks*)

Solution:

Suppose your birthday is Jan. 1st. Then none of the n people sampled randomly can have birthday on Jan. 1st. Hence there is 364^n possible ways for these n. On the other hand the total number of ways for their birthdays is 365^n. Hence the chance is of having none of the n people sharing your birthday is

$$\left(\frac{364}{365}\right)^n.$$

- (b) How many people do we need to select so that the probability is at least .5 that at least one shares your birthday? (*4 marks*)

Solution:

Let $A = \{\text{having at least one shares your birthday}\}$. We note that $A^c = \{\text{no one sharing your birthday}\}$. Thus

$$P(A) = 1 - P(A^c) = 1 - \left(\frac{364}{365}\right)^n.$$

We then need to choose n such that $P(A) \geq 0.5$. This then implies that

$$n \geq \frac{\log 2}{\log(1 + \frac{1}{364})} \approx 252.65.$$

Question 2. Two methods, A and B, are available for teaching a certain industrial skill. The failure rate is 20% for A and 10% for B. However, B is more expensive and hence is used only 30% of the time. (A is used the other 70%.) A worker was taught the skill by one of the methods but failed to learn it correctly. What is the probability that she was taught by method A?(*6 marks*)

Solution:

Let $A = \{\text{method A being used}\}$, $B = \{\text{method B being used}\}$ and $F = \{\text{Failing to learn the method properly}\}$. We then have $P(A) = .7$, $P(B) = .3$, $P(F|A) = .2$, while $P(F|B) = .1$. We need to find $P(A|F)$. Using Bayes' theorem we have

$$\begin{aligned} P(A|F) &= \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|B)P(B)} \\ &= \frac{(.2) \times (.7)}{(.2) \times (.7) + (.1) \times (.3)} \\ &= \frac{14}{17} \approx .82. \end{aligned}$$

Question 3. A particular sale involves four items randomly selected from a large lot that is known to contain 10% defective items. Let Y denote the number of defective items among the four sold. The purchaser of the items will return the defective items for repair, and the repair cost is given by $C = 3Y^2 + Y + 2$. Find the expected repair cost. (*6 marks*)

Solution:

We note that $Y \sim Bin(n = 4, p = .1)$. Then $\mathbb{E}(Y) = np = 4 \times (.1) = .4$ and $\mathbb{E}(Y^2) = Var(Y) + [\mathbb{E}(Y)]^2 = np(1-p) + n^2p^2 = .36 + .16 = .52$. On the other hand,

$$\mathbb{E}(C) = 3\mathbb{E}(Y^2) + \mathbb{E}(Y) + 2 = 3 \times (.52) + .4 + 2 = 3.96.$$

Question 4. A food manufacturer uses an extruder (a machine that produces bite-size cookies and snack food) that yields revenue for the firm at a rate of $200 per hour when in operation. However, the extruder breaks down an average of two times every day it operates. Suppose Y, the number of break downs per day, is distributed according to a Poisson distribution. If the daily revenue generated by the machine is $R = 1600 - 50Y^2$. Find the expected daily revenue for the extruder. (*6 marks*)

Solution:

We have $Y \sim Po(\lambda = 2)$. Then $\mathbb{E}(Y) = Var(Y) = \lambda = 2$, and therefore $\mathbb{E}(Y^2) = Var(Y) + [\mathbb{E}(Y)]^2 = \lambda + \lambda^2 = 2 + 2^2 = 6$. On the other hand,

$$\mathbb{E}(R) = 1600 - 50\mathbb{E}(Y^2) = 1600 - 50 \times (6) = 1300.$$