

# MATH2004 C — Test 2: 11:35 am - 12:25 pm, Oct. 26

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

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1. (1.5 marks) Find the slope of the equation of the tangent line to  $r = 4 \cos \theta$  at  $\theta = \pi/3$ .

- (a) 1 (b) 2 (c) 3 (d)  $\sqrt{3}$  (e)  $\frac{1}{\sqrt{3}}$

**Solution:** (e)

$$\frac{dy}{dx} = \frac{r'_\theta \sin \theta + r \cos \theta}{r'_\theta \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos^2 \theta}{-2 \sin \theta \cos \theta} = \frac{1}{\sqrt{3}}.$$

2. (1.5 marks) Given the linear equation of the plane  $\Pi: 2x + 3y + 4z - 4 = 0$  and the line  $L: \mathbf{r} = (1, 2, 3) + t(3, 2, -1)$ , find the intersection between the plane  $\Pi$  and the line  $L$ .

- (a) (5, -2, 5) (b) (-5, 2, 5) (c) (-5, -2, -5) (d) (5, 2, 5) (e) (-5, -2, 5)

**Solution:** (e)

The parametric equation of the line is:

$$x = 1 + 3t, y = 2 + 2t, z = 3 - t.$$

Substitute this into the plane we get

$$2(1 + 3t) + 3(2 + 2t) + 4(3 - t) - 4 = 0, \Rightarrow t = -2.$$

Thus the point is (-5, -2, 5).

3. (1.5 marks) Let  $\mathbf{r}(t) = (6t, 8t, te^t) = 6t\mathbf{i} + 8t\mathbf{j} + te^t\mathbf{k}$ . Calculate  $\int_0^1 \mathbf{r}(t) dt$ .

- (a) (3, 4, e) (b) (3, 4, 1) (c) (6, 8, e) (d) (6, 8, 1) (e) (3, 4, -e)

**Solution:** (b).

$$\int_0^1 \mathbf{r}(t) dt = \left( \int_0^1 6t dt, \int_0^1 8t dt, \int_0^1 te^t dt \right) = (3, 4, 1).$$

4. (1.5 marks) Let  $\mathbf{r}(t) = (4t - 1, t^2, t^3 - 3t) = (4t - 1)\mathbf{i} + t^2\mathbf{j} + (t^3 - 3t)\mathbf{k}$ . Find the curvature  $\kappa(0)$  of  $\mathbf{r}(t)$  at  $t = 0$ .

- (a)  $\frac{2}{25}$  (b)  $\frac{6}{5}$  (c)  $\frac{4}{5\sqrt{5}}$  (d) 2 (e) 7

**Solution:** (a).

$\mathbf{r}'(t) = (4, 2t, 3t^2 - 3)$ ,  $\mathbf{r}''(t) = (0, 2, 6t)$ . At  $t = 0$ ,  $\mathbf{r}'(0) = (4, 0, -3)$ ,  $\mathbf{r}''(0) = (0, 2, 0)$ ,  $\mathbf{r}'(0) \times \mathbf{r}''(0) = (6, 0, 8)$ . Hence

$$|\mathbf{r}'(0)| = 5, |\mathbf{r}'(0) \times \mathbf{r}''(0)| = 10.$$

The curvature at  $t = 0$  is:

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{10}{5^3} = \frac{2}{25}.$$

5. (6 points) Let  $\Pi$  be the plane passing through the three points  $P(1, 2, -1)$ ,  $Q(3, 1, -1)$  and  $R(1, 1, 0)$ .

(i) (4 points) Find the linear equation of the plane  $\Pi$ .

(ii) (2 points) Find the distance from the point  $S(8, 1, 1)$  to the plane  $\Pi$ .

**Solution:** (i) We have

$$\vec{PQ} = \langle 2, -1, 0 \rangle, \quad \vec{PR} = \langle 0, -1, 1 \rangle.$$

The normal vector of the plane is

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = \langle -1, -2, -2 \rangle.$$

The equation of the plane is

$$\langle -1, -2, -2 \rangle \cdot (\mathbf{r} - \langle 1, 2, -1 \rangle) = 0, \Rightarrow -(x - 1) - 2(y - 2) - 2(z + 1) = 0,$$

i.e.,

$$x + 2y + 2z - 3 = 0.$$

(ii). The distance is:

$$d = \frac{|8 + 2(1) + 2(1) - 3|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$

6. (8 points) Let  $C$  be  $\mathbf{r}(t) = (3t, \cos(4t), \sin(4t)) = 3t\mathbf{i} + \cos(4t)\mathbf{j} + \sin(4t)\mathbf{k}$ .

(i)(3 points) Find the length of the curve from  $t = 0$  to  $t = 2$ .

(ii) (5 points = 2+3) Find the unit tangent vector  $\mathbf{T}(t)$  and the unit normal vector  $\mathbf{N}(t)$ .

**Solution:** (i)  $\mathbf{r}'(t) = \langle 3, -4\sin(4t), 4\cos(4t) \rangle$ , which gives  $|\mathbf{r}'(t)| = 5$ .

So the length is

$$L = \int_0^2 |\mathbf{r}'(t)| dt = \int_0^2 5 dt = 10.$$

(ii) From (i) we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle 3/5, -4/5 \sin(4t), 4/5 \cos(4t) \rangle .$$

$$\mathbf{T}'(t) = \langle 0, -16/5 \cos(4t), -16/5 \sin(4t) \rangle, \quad |\mathbf{T}'(t)| = 16/5.$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle 0, -\cos(4t), -\sin(4t) \rangle .$$