Sample Midterm II- English translation.

Problem 1: __________

Problem 2: __________

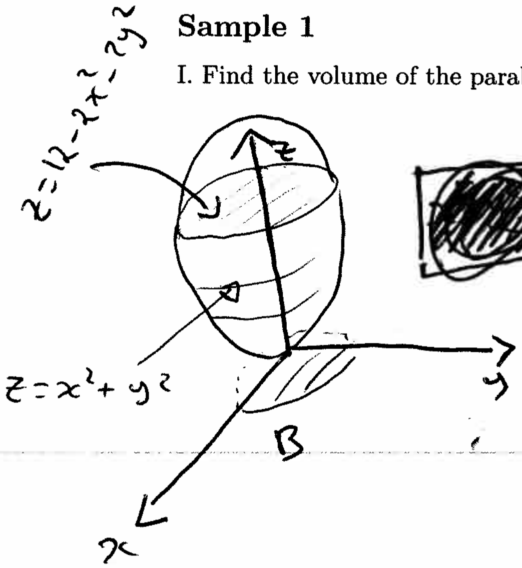
Problem 3: __________

Problem 4: __________

Total: __________

Sample 1

I. Find the volume of the paraboloid enclosed by the surfaces $z = 12 - 2x^2 - 2y^2$ and $z = x^2 + y^2$



$$D = \left\{ (x, y, z) \in B, \text{ AND } x^2+y^2 \leq z \leq 12 - 2x^2 - 2y^2 \right\}$$

TO FIND B: INTERSECT THE GIVEN SURFACES:

$$12 - 2x^2 - 2y^2 = x^2 + y^2 \longrightarrow x^2 + y^2 = 4$$

HENCE $B = \{ x^2 + y^2 \leq 4 \}$. ($B$ IS THE PROJECTION ONTO THE xy PLANE OF THE DISK BOUNDED BY THE INTERSECTION OF THE 2 SURFACES)

THE SOLID HAS ROTATIONAL SYMMETRY. LET'S USE POLAR COORDINATES TO DESCRIBE B

$$B = \{ (r, \theta) \mid 0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 2 \}$$

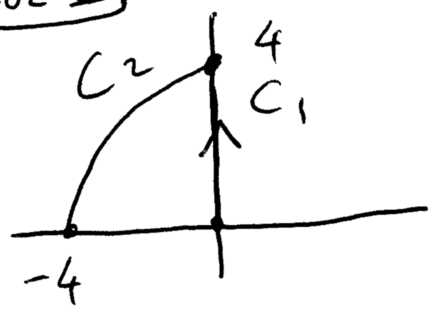
$$\mathrm{Vol}(D) = \iint_B \int_{x^2+y^2}^{12-2x^2-2y^2} dz\, dA \underset{\text{USE } x^2+y^2=r^2}{=}$$

$$\int_0^{2\pi} \int_0^2 ((12 - 2r^2) - r^2) r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (12r - 3r^3)\, dr\, d\theta =$$

$$2\pi \left(6r^2 - \frac{3}{4} r^4 \right)\Big|_0^2 = 24\pi$$

II. Calculate the line integral of $\mathbf{F} =< 4x, 2-2y >$, along the curve C, which consists of the segment from $(0,0)$ to $(0,4)$ and the arc of the circle $x^2+y^2=16$ from $(0,4)$ to$(-4,0)$, oriented counterclockwise.

SOL 1)



$$C = C_1 + C_2$$

$$C_1 : \{ \vec{r}_1(t) = (0, t) \quad 0 \le t \le 4 \}$$

$$C_2 = \{ \vec{r}_2(t) = (4\cos t, 4\sin t) \quad \tfrac{\pi}{2} \le t \le \pi \}$$

$$\int_C \vec{F} d\vec{r} = \int_{C_1} \vec{F} d\vec{r}_1 + \int_{C_2} \vec{F} d\vec{r}_2 = \int_0^4 \langle 0, 2-2t\rangle \cdot \langle 0, 1\rangle \, dt$$

$$+ \int_{\pi/2}^{\pi} \langle 16\cos t, 2 - 8\sin t\rangle \langle -4\sin t, 4\cos t\rangle \, dt =$$

$$= \int_0^4 (2-2t)\, dt + \int_{\pi/2}^{\pi} (-96\cos t \sin t + 8\cos t)\, dt =$$

$$2t - t^2 \Big|_0^4 + \left(24\cos(2t) + 8\sin t\right)\Big|_{\pi/2}^{\pi} =$$

$$-8 + 40 = 32$$

NOTE) YOU CAN ALSO PARAMETRIZED ~~[illegible]~~ C_1 BY $\vec{r}_1(t) = (0, 4t)$ WITH $0 \le t \le 1$
(COMPARE THE 2 PARAMETRIZATION TO UNDERSTAND)

SOLUTION USING FUNDAMENTAL THEOREM

II. Calculate the line integral of $\mathbf{F} =< 4x, 2-2y >$, along the curve C, which consists of the segment from $(0,0)$ to $(0,4)$ and the arc of the circle $x^2+y^2 = 16$ from $(0,4)$ to$(-4,0)$, oriented counterclockwise.

SOL 2 OBSERVE THAT $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \; (=0)$.

SINCE $\vec{F}$ HAS A SIMPLY CONNECTED DOMAIN, $\vec{F}$ IS CONSERVATIVE $(\vec{F} = \nabla f)$

TO FIND $\vec{F}$

$$\frac{\partial f}{\partial x} = 4x \Rightarrow f = \underbrace{2x^2 + K(y)}$$

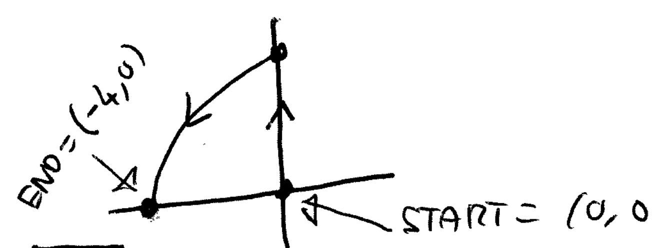
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(2x^2 + K(y)\right) = K'(y) = 2-2y$$

$$\Rightarrow K(y) = 2y - y^2.$$

$$\Rightarrow f = 2x^2 + 2y - y^2.$$

HENCE, BY THE FUNDAMENTAL THM OF LINE INTEGRALS,

$$\int_C \vec{F}\, d\vec{r} = f(-4,0) - f(0,0) = 32$$



III. Only one of the following vector fields is conservative. Find it and calculate its potential.

1. $\mathbf{F} = < xy^3, x^2y >$
2. $\mathbf{G} = < 2x - \cos y, x \sin y >$
3. $\mathbf{H} = < \cos y, \sin x >$

WE ASSUME D = THE WHOLE xy-PLANE, WHICH IS SIMPLY CONNECTED. HENCE WE JUST NEED TO CHECK IF $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

1. $\frac{\partial F_1}{\partial y} = 3xy^2 \neq \frac{\partial F_2}{\partial x} = 2xy$ NOT CONSERVATIVE

2. $\frac{\partial G_1}{\partial y} = \sin y = \frac{\partial F_2}{\partial x}$ CONSERVATIVE

3. $\frac{\partial H_1}{\partial y} = -\sin y \neq \frac{\partial H_2}{\partial x} = \cos x$ NOT CONSERVATIVE

POTENTIAL f FOR G $(\vec{G} = \nabla f)$

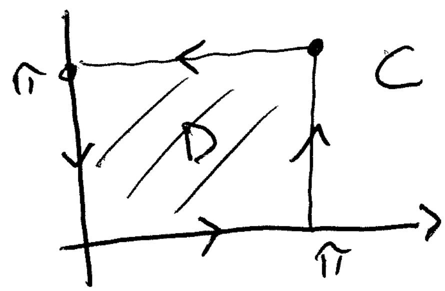
$$\text{I} \begin{cases} \frac{\partial f}{\partial x} = 2x - \cos y \implies f = x^2 - x\cos(y) + K(y) \\ \Downarrow \\ \frac{\partial f}{\partial y} = x\sin y \implies \frac{\partial f}{\partial y} = x\sin(y) + K'(y) = x\sin y \end{cases} \text{II}$$

$\Downarrow$

$K' = 0 \quad K = \text{CONST}$

HENCE $f = x^2 - x\cos y + K$

IV. Let C be the square with vertices $(0,0)$, $(\pi,0)$, (π,π), $(0,\pi)$, oriented counterclockwise. Using Green's Theorem, calculate the line integral of the vector field $\mathbf{F} = < x\cos y, 2y\sin x >$



C = UNION OF 4 SEGMENTS

D = SQUARE BOUNDED BY C

$$\int_C \vec{F}\, d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA =$$

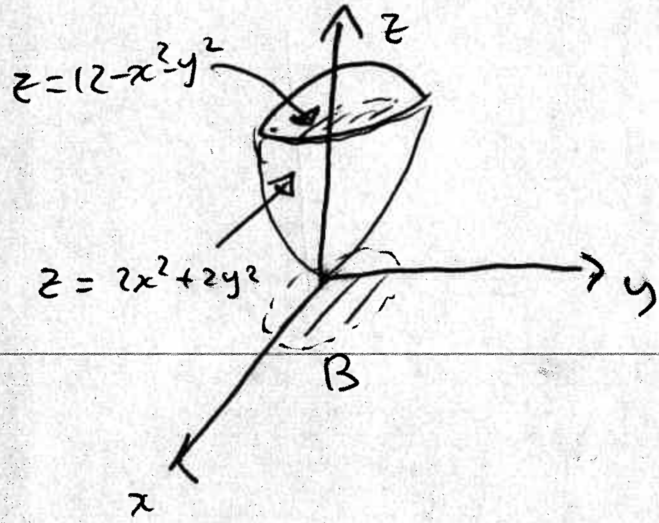
$$= \int_0^{\pi} \int_0^{\pi} \left(2y\cos x + x\sin(y) \right) dx\, dy =$$

$$= \int_0^{\pi} \left(2y\sin x + \frac{x^2}{2}\sin(y) \right) \Big|_0^{\pi} dy = \int_0^{\pi} \frac{\pi^2}{2}\sin(y)\, dy$$

$$= -\frac{\pi^2}{2}\cos(y) \Big|_0^{\pi} = \pi^2$$

Sample 2

I. Find the volume of the paraboloid enclosed by the surfaces $z = 12 - x^2 - y^2$ and $z = 2x^2 + 2y^2$



$$D = \left\{ (x,y,z) \mid (x,y) \in B \quad 2x^2+2y^2 \le z \le 12-x^2-y^2 \right\}$$

TO FIND B: AS IN SAMPLE I, INTERSECT THE GIVEN SURFACES:

$$2x^2+2y^2 = 12 - x^2 - y^2$$

$$r^2 \quad x^2+y^2 = 4$$

IN POLAR COORDINATES: $B = \{ 0 \le \theta \le 2\pi, \ 0 \le r \le 2 \}$

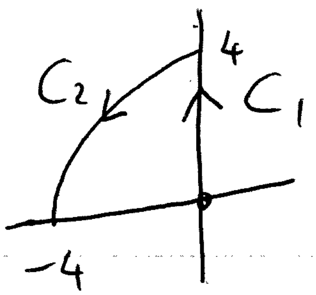
$$Vol(D) = \iint_B (12 - x^2 - y^2) - (2x^2+2y^2)\, dA \underset{\text{USE } x^2+y^2=r^2}{=}$$

$$= \int_0^{2\pi}\int_0^2 \left((12 - r^2) - (2r^2)\right) r\, dr\, d\theta$$

$$= 2\pi \int_0^2 (12r - 3r^3)\, dr = 24\pi$$

II. Calculate the line integral of $\mathbf{F} = < 2x, 1-2y >$, along the curve C, which consists of the segment from $(0,0)$ to $(0,4)$ and the arc of the circle $x^2+y^2=16$ from $(0,4)$ to $(-4,0)$, oriented counterclockwise.

SOL 1 SAME SET-UP OF SAMPLE I



$C = C_1 + C_2$

C_1 PARAMETRIZED BY $r_1(t) = (0, t)$ $\quad 0 \le t \le 4$

C_2 PARAMETRIZED BY

$r_2(t) = (4\cos t, 4\sin t) \quad \frac{\pi}{2} \le t \le \pi$

$$\int_C \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r}_1 + \int_{C_2} \vec{F}\, d\vec{r}_2 =$$

$$\int_0^4 \langle 0, 1-2t \rangle \cdot \langle 0, 1 \rangle \, dt + \int_{\pi/2}^{\pi} \langle 2\cdot 4\cos t, 1 - 2\cdot 4\sin t \rangle \cdot$$

$$\cdot \langle -4\sin t, 4\cos t \rangle \, dt = \int_0^4 (1-2t)\, dt +$$

$$+ \int_{\pi/2}^{\pi} (-64\cos t \sin t + 4\cos t)\, dt = (t - t^2)\Big|_0^4$$

$$+ \left(16\cos(2t) + 4\sin t\right)\Big|_{\pi/2}^{\pi} = 28$$

SOL 2) OF II. (USING FUNDAMENTAL THEOREM)

$$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x} \Rightarrow \vec{F} \text{ IS CONSERVATIVE}$$

LET'S FIND THE POTENTIAL: $\nabla f = \vec{F}$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x \Rightarrow f = \underbrace{x^2 + k(y)} \\ \frac{\partial f}{\partial y} = 1 - 2y \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + k(y)) \end{cases}$$

$$= k'(y) = 1 - 2y$$

$$k = y - y^2 + c$$

$$f = x^2 + y - y^2 + c$$

$$\Rightarrow \int_C \vec{F}\, d\vec{r} = f(-4, 0) - f(0, 0) = 16$$

$\boxed{7 \text{ BIS}}$

III. Only one of the following vector fields is conservative. Find it and calculate its potential.

1. $\mathbf{F} =< \sin y, \cos y >$
2. $\mathbf{G} =< xy^2, -x^2 >$
3. $\mathbf{H} =< 2x + \sin y, x \cos y >$

1. $\dfrac{\partial F_1}{\partial y} = \cos y \neq \dfrac{\partial F_2}{\partial x} = 0$ NOT CONSERVATIVE

2. $\dfrac{\partial G_1}{\partial y} = y^2 \neq \dfrac{\partial G_2}{\partial x} = -2x$ NOT CONSERVATIVE

3. $\dfrac{\partial H_1}{\partial y} = \cos(y) = \dfrac{\partial H_2}{\partial x} = \cos y$

$\vec{H}$. IS CONSERVATIVE. TO FIND f:

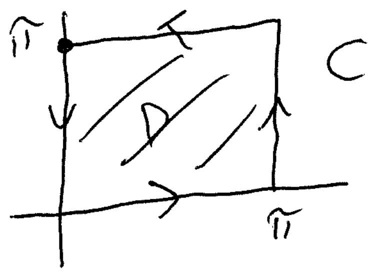
$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + \sin(y) \Rightarrow f = \underbrace{x^2 + x\sin(y) + k(y)} \\ \dfrac{\partial f}{\partial y} = x\cos y \Longrightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(x^2 + x\sin y + k(y)\right) \end{cases}$$

$$= x\cos y + k'(y).$$

$\Rightarrow k' = 0 \quad k = \text{CONSTANT}$

$f = x^2 + x\sin(y) + k$

IV. Let C be the square with vertices $(0,0)$, $(\pi,0)$, (π,π), $(0,\pi)$, oriented counterclockwise. Using Green's Theorem, calculate the line integral of the vector field $\mathbf{F} =< x\cos y, y\sin x >$



$$\int_C \vec{F}\,d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA =$$

$$\int_0^{\pi}\int_0^{\pi} (y\cos x + x\sin y)\,dy\,dx =$$

$$\int_0^{\pi} \left(\left(\frac{y^2}{2}\cos x - x\cos y\right)\right)\Bigg|_0^{\pi} dx = \int_0^{\pi} \left(\frac{\pi^2}{2}\cos x + 2x\right) dx$$

$$= \left(\frac{\pi^2}{2}\sin(x) + x^2\right)\Bigg|_0^{\pi} = \pi^2$$