MAT 1341A.F11 Test 2 Solutions

1. For what value of α is the set of vectors $\underbrace{\{(1,1,1),(1,2,0),\ (2,3,\alpha)\}}_{S}$ linearly dependent?

A. -1
B. 2
C. 0
D. 1
E. -1/2
F. -2

S is linearly dependent $\Leftrightarrow$ rank $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & \alpha \end{bmatrix} < 3.$

But rank $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & \alpha \end{bmatrix}$ = rank $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & \alpha-2 \end{bmatrix}$

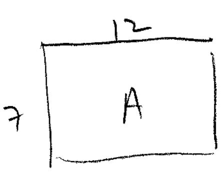
= rank $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \alpha-1 \end{bmatrix} < 3 \Leftrightarrow \alpha - 1 = 0 \Leftrightarrow \alpha = 1$

2. If A is a 7×12 matrix, what is the smallest possible dimension of the kernel of A?

A. 0
B. 3
C. 5
D. 7
E. 11
F. 12

dim ker A + rank A = # cols of A

dim ker A + rank A = 12



But rank $A \le \min(7,12) = 7$

(so $-$rank $A \ge -7$)

$\therefore$ dim ker A = 12 $-$ rank A

$\ge 12 - 7 = 5.$

3. If $B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, then the second row of B^{-1} is:

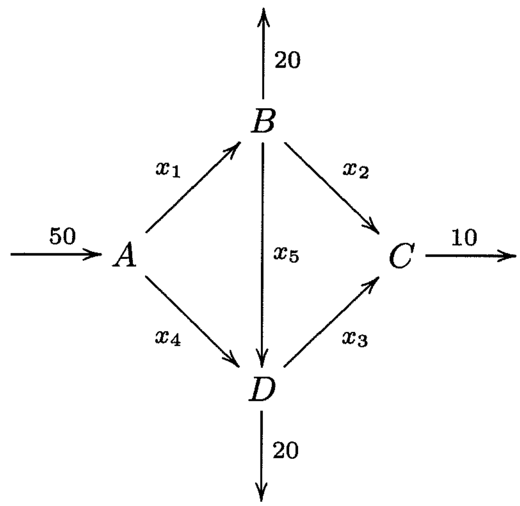
A. $[1 \ 0 \ -1]$
B. $[-1 \ 0 \ 1]$
C. $[0 \ 1 \ -1]$
D. $[2 \ 0 \ -1]$
E. $[1 \ -1 \ 0]$
F. None of the above

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} & * & & & * & \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] = [A|C]$$

Further operations to take $[A|C]$ to I_3 will not change the second row of C. Hence the second row of B^{-1} is $[-1 \ 0 \ 1]$

4. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter A or leave B,C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



[2] a) Write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Intersection	Flow In	=	Flow Out
A	50	$=$	$x_1 + x_4$
B	x_1	$=$	$20 + x_2 + x_5$
C	$x_2 + x_3$	$=$	10
D	$x_4 + x_5$	$=$	$20 + x_3$

$4 @ \frac{1}{2}$ each

Constraints: $x_i \geq 0$, $i = 1, \ldots, 4$ (one-way streets) $\frac{1}{2}$

$x_i \in \mathbb{Z}$ (Integers) (no fractional cars.) $\frac{1}{2}$

[2] 4b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{c} \begin{array}{ccccc|c} & & & s & t & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 1 & 1 & 30 \\ 0 & 0 & 1 & -1 & -1 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= 50 - s \\ x_2 &= 30 - s - t \\ x_3 &= -20 + s + t \qquad ; \quad s, t \in \mathbb{R} \\ x_4 &= s \\ x_5 &= t \end{aligned}$$

$4 @ \frac{1}{2}$ each $= 2$

(allow one error)

[2] c) If $\overline{BD}$ is closed find the maximum and minimum flows along $\overline{BC}$

$\boxed{\text{Min} = \tfrac{1}{2} \quad \text{Max} = \tfrac{1}{2} \quad \text{Justif}^{n}\ 1}$

Road $\overline{BD}$ is closed $\Leftrightarrow$ $x_5 = 0$ $\Leftrightarrow$ $t = 0$.

Hence
$$\begin{aligned} x_1 &= 50 - s \\ x_2 &= 30 - s \\ x_3 &= -20 + s \\ x_4 &= s \end{aligned}$$

Applying the first constraints, we find

$x_1 \geq 0 \Leftrightarrow 50 - s \geq 0 \Leftrightarrow 50 \geq s$

$x_2 \geq 0 \Leftrightarrow 30 - s \geq 0 \Leftrightarrow 30 \geq s$

$x_3 \geq 0 \Leftrightarrow -20 + s \geq 0 \Leftrightarrow s \geq 20$

$x_4 \geq 0 \Leftrightarrow s \geq 0$.

Thus $30 \geq s \geq 20$.

Hence x_2, the flow along $\overline{BC}$ is smallest when s is largest ie. $s = 30$. Then, $x_2 = 0$ is the minimum flow

The maximum flow along $\overline{BC}$ is $30 - 20 = \underline{10}$

5. Let $W = \text{span}\{(1,0,1,1),(-1,1,2,0),(1,1,4,2),(0,1,3,1)\}$, and define a matrix A by

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

$1\frac{1}{2}$ a) Find a basis for W which is a **subset** of the given spanning set above.

$1\frac{1}{2}$ b) Extend your basis for W in part (a) to a basis of $\mathbf{R}^4$.

$1\frac{1}{2}$ c) Find a basis for $\ker A$, and hence find its dimension.

$1\frac{1}{2}$ d) Extend your basis of $\ker A$ in (c) to a basis of $\mathbf{R}^4$.

$$A \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\text{(for (c))}}{\sim} \begin{array}{c} \begin{array}{cccc} & & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad (*)$$

a) By the columnspace algorithm, since $W = \text{col}(A)$, we know $\{\underset{v_1}{(1,0,1,1)}, \underset{v_2}{(-1,-1,2,0)}\}$ is a basis for W (and is a subset of the given spanning set).

1 – any correct answer
1/2 – justn

1/2 – any correct extension
1 – justn

b) Note that if $v_3, v_4 \in \mathbb{R}^4$, $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ & v_3 & & \\ & v_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ & v_3 & & \\ & v_4 & & \end{bmatrix}$; Hence if $v_3 = (0,0,1,0)$, $v_4 = (0,0,0,1)$, rank $B = 4$ and so $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$ which extends the basis $\{v_1, v_2\}$ of W.

c) From $(*)$, we see $\ker A = \{(-2s-t, -s-t, s, t) \mid s,t \in \mathbb{R}\}$ 1

$= \text{span}\{(-2,-1,1,0),(-1,-1,0,1)\}$. Hence $\{\underset{u_1}{(-2,-1,1,0)}, \underset{u_2}{(-1,-1,0,1)}\}$ is a basis for $\ker A$. Thus $\dim \ker A = 2$ 1/2

5(d) Note if $u_3, u_4 \in \mathbb{R}^4$,

$$C = \begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ & u_3 & & \\ & u_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ & & u_3 & \\ & & u_4 & \end{bmatrix}$$

Hence if $u_3 = (0,0,1,0)$ & $u_4 = (0,0,0,1)$,

rank $C = 4$ and so $\{u_1, u_2, u_3, u_4\}$ is a basis of $\mathbb{R}^4$ which extends the basis $\{u_1, u_2\}$ of ker A

(1½) – any correct extension
(1) – justn

6. Suppose A is an $n \times n$ matrix and that,

a, b, c : (1) - correct answer
(1) - justification

there is a vector $b \in \mathbf{R}^n$, for which $Ax = b$ is inconsistent.

State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example (with numbers!).
- If you say the statment is true, you must give a clear explanation.

*Since $Ax=b$ is not consistent for all $b \in \mathbb{R}^n$, A is not invertible

a) The system $Ax = 0$ has a unique solution.

$\boxed{\text{FALSE}}$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $[A|b] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$ is inconsistent, and here,

$\ker A = \text{span}\{(0,1)\}$, so $Ax=0$ has ∞ly many solutions

b) The matrix A is not invertible.

$\boxed{\text{TRUE}}$

(See * above) This follows from a theorem in class: A $n \times n$ invertible $\Leftrightarrow$ ($Ax=b$ is consistent $\forall b \in \mathbb{R}^n$).

c) The rows of A are linearly independent.

The same example as in part (a) works. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\boxed{\text{FALSE}}$

Then $[A|b]$ is inconsistent and $\{(1,0),(0,0)\}$ is dependent because it contains the zero vector.

Alternatively, we know rank $A = 1 < 2$, so the rows of A are dependent. (But this is obvious in our example!)

[4] **7 [Bonus].** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose that $ad - bc = 0$. Show carefully that A is not invertible.

Suppose $ad-bc=0$, and that A __is__ invertible. (We seek a contradiction.).

First note that $A \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since A is invertible, $A^{-1}\left(A \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = A^{-1} \cdot 0 = 0$

i.e. $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $d = b = c = a = 0$. But then $A = 0$,

and 0 is certainly not an invertible matrix!

[Note that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so if $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^{-1} = D$ existed,

$$\underbrace{D \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{I} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = D \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, a contradiction.]

1 – Some correct idea
+1 – Some " progress
+1 – substantially correct
+1 – correct and well-written