[*5 marks*] 1. Define the term *incomplete market* and list three features (not including the defining feature) of an incomplete market.

An incomplete market is one where not all payoffs are attainable; i.e. the set of attainable payoffs (column space of the payoff matrix) is a strict subset of the set of imaginable payoffs ($\mathbb{R}^M$). Features include but are not limited to

- *Non-unique replicating portfolios for attainable payoffs.*
- *Non-unique state price vectors.*
- *The rank of the payoff matrix is strictly less than the number of states of nature.*
- *The risk associated with selling derivatives cannot be completely hedged.*
- *It is not necessarily possible to "pick your payoff" in every state of nature.*

[*5 marks*] 2. Define the term *state price vector* and carefully explain the relationship between (i) the sum of the state prices and (ii) the risk-free rate of interest in a market.

A state price vector is a $1 \times M$ vector Ψ that solves the system $D \cdot \Psi = \mathbf{S}_0$. The elements of Ψ can be interpreted as the price of a state i dollar, i.e. the price of a security that pays one dollar in state i. The relationship between the sum of the state prices and the risk-free rate is

$$\sum_{i=1}^{M} \Psi_i = \frac{1}{1+r} ,$$

where r is the risk-free rate of interest and Ψ_i is the i^{th} component of Ψ (i.e. the price of a state i dollar). This makes sense because the left-hand side is the cost of a portfolio that will be worth exactly one dollar regardless of which state of nature is realized, and can therefore be interpreted as the present value of one dollar (i.e. the amount that must be set aside today in order to ensure one dollar in the future).

[*4 marks*] 3. Define the term *arbitrage portfolio.*

A portfolio ϕ that is either a free lottery ticket, i.e.

$$\phi \cdot \mathbf{S}_0 = 0 , \qquad \phi \cdot D > 0 ,$$

or a free two-dollar bill, i.e.

$$\phi \cdot \mathbf{S}_0 < 0 , \qquad \phi \cdot D \geq 0 .$$

[*6 marks*] 4. Determine whether or not there exist any arbitrage opportunities in the following market

$$D = \begin{bmatrix} 10 & 15 \\ 10 & 20 \end{bmatrix} , \qquad \mathbf{S}_0 = \begin{bmatrix} 12 \\ 16 \end{bmatrix} .$$

You do not need to identify/construct an arbitrage portfolio, but if you do there will be a small number of bonus marks in it for you.

It suffices to determine whether or not there exists a strictly positive state price vector. To this end note that the unique solution to the system

$$\begin{aligned} 10\Psi_1 + 15\Psi_2 &= 12 \\ 10\Psi_1 + 20\Psi_2 &= 16 \end{aligned}$$

is $(\Psi_1, \Psi_2) = (0, 0.8)$. The easiest way to see this is to subtract one equation from the other to find that $5\Psi_2 = 4$, or $\Psi_2 = 0.8$, and then plug the result back into either equation and solve for Ψ_1. Since this vector is not strictly positive, the market admits arbitrage.

5. State whether or not you agree, partially agree or disagree with each of the following statements. In each case justify your answer (relatively briefly).

[*5 marks*] (a) Actual/real-world probabilities don't matter at all for the pricing and risk-management of derivatives.

What I was looking for here was something along the lines of "risk-neutral probabilities *are used for pricing (they enter the derivative valuation formula) but actual probabilities are used for risk-management (for example they are used to compute the risk-minimizing hedge ratio in the trinomial model)." I received many intelligent answers that differed from this and awarded either full or part marks accordingly (and more or less subjectively, see me in person if you'd like more details).*

[*5 marks*] (b) The average profit from selling naked (i.e. unhedged) put options is positive.

In the context of the binomial model this statement is true, as proven on the second assignment (there we proved that the expected return from buying the put is negative, which means that the expected return from selling is positive). As in (a) I got lots of intelligent answers that were not exacly along this line, and awarded full/part marks subjectively (see me in person if you'd like to vent). Several students referred to the "picking up pennies in front of a steam-roller" strategy (glad to see people paying attention) - this is an example of a strategy with a very small positive expected return but MASSIVE volatility (the "double-down" strategy where you start by betting one dollar and either (i) double your bet if you lose or (ii) quit the first time you win a bet is another example). So strictly speaking, even in the real world the return to naked puts is positive, but entails a gigantic (but unlikely) downside. You would need a tremendous amount of capital (used to absorb the huge but rare losses) in order to generate a profit in the long run.

[*5 marks*] (c) Both the value and hedge ratio of a call option will rise if the volatility of the underlying stock rises.

Here I was looking for a reference to Question 3 on the second assignment, where the value of the call rose but the value of the hedge ratio could either rise or fall depending on moneyness of the option. Many people made reference to the optimal hedge ratio $\delta^* = \rho\,\sigma_f/\sigma_S$ *that we discussed in class and mentioned that if* σ_S *increases then* δ^* *will decrease. This is true provided* σ_f *does not change when* σ_S *changes - however it is almost always the case that the volatility of the derivative payoff will increase with the volatility of the underlying which means the overall impact on* δ^* *is ambigious.*
In lecture, when using our intuition to understand this formula, we said something along the lines of "all else being equal, if σ_S *is very large then* δ^* *will be very small which makes sense, since you won't use an extremely volatile instrument to hedge risk." The "all else being equal" here is critical, but since we didn't make a huge deal about all else being equal in lecture, I tried not to be too punitive here.*

[*15 marks*] 6. Consider a trinomial market with payoff matrix and initial price vector

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 95 & 100 & 120 \end{bmatrix}, \qquad \mathbf{S}_0 = \begin{bmatrix} 1 \\ 100 \end{bmatrix}.$$

(a) Verify that if $\Psi_m \in \mathbb{R}$ and

$$\Psi = [0.8 - 0.8\Psi_m \qquad \Psi_m \qquad 0.2 - 0.2\Psi_m]^T,$$

then Ψ is a state price vector.

It suffices to confirm that $D \cdot \Psi = \mathbf{S}_0$. *Carrying out the required multiplication we get*

$$\begin{bmatrix} 1 & 1 & 1 \\ 95 & 100 & 120 \end{bmatrix} \cdot \begin{bmatrix} 0.8 - 0.8\Psi_m \\ \Psi_m \\ 0.2 - 0.2\Psi_m \end{bmatrix} = \begin{bmatrix} 0.8 - 0.8\Psi_m + \Psi_m + 0.2 - 0.2\Psi_m \\ 76 - 76\Psi_m + 100\Psi_m + 24 - 24\Psi_m \end{bmatrix} = \begin{bmatrix} 1 \\ 100 \end{bmatrix},$$

as required.

(b) If a call option struck at \$100 is introduced to this market at a price of \$3.75, does the market remain free of arbitrage?

The market will remain free of arbitrage provided the call is introduced at a price of the form $c_u\Psi_u + c_m\Psi_m + c_d\Psi_d$, *where* $[\Psi_u \quad \Psi_m \quad \Psi_d]$ *is any strictly positive state price vector and* c_i *is the payoff of the call in state* i. *Using the general form of the state price vector from (a) and the fact that* $c_u = 20$, $c_m = c_d = 0$ *this simplifies to* $20 \cdot (0.2 - 0.2\Psi_m) = 4 - 4\Psi_m$. *If the call is priced at \$3.75 the implied middle-state price is* $\Psi_m = (4 - 3.75)/4 = 0.0625$, *which yields* $\Psi_d = 0.75$ *and* $\Psi_u = 0.1875$. *This*

corresponds to a valid (i.e. strictly positive) state price vector and so the answer is yes, the market remains free of arbitrage.
Alternatively we could observe from (a) that the arbitrage-free range for the middle-state price is $\Psi_m \in (0,1)$. *Since the price of the call is* $4 - 4\Psi_m$, *the arbitrage-free range for the call price is* $(0,4)$. *Since $3.75 lies in this range, the market does remain free of arbitrage if the call is introduced at this price.*
Finally, we could look at the new/enlarged market defined by

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 95 & 100 & 120 \\ 0 & 0 & 20 \end{bmatrix}, \qquad \mathbf{S}_0 = \begin{bmatrix} 1 \\ 100 \\ 3.75 \end{bmatrix}.$$

and confirm that it is free of arbitrage.

(c) What is the maximum price you would expect your clients (who are reasonable people) to pay for the option in (b)?
The reasonable/risk-averse range for Ψ_m *is* $(0.2/1.2, 0.8/1.8) = (0.166\ldots, 0.444\ldots)$, *so the maximum price I would expect them to pay is* $4 - 4(0.2/1.2) = 3.33\ldots$.

[*5 marks*] 7. Consider an incomplete Arrow-Debreu market with payoff matrix D. Suppose that $\mathbf{v}$ is an attainable payoff and let $\phi^{(1)}$, $\phi^{(2)}$ be any two portfolios that replicate $\mathbf{v}$. Without reference to the fundamental theorem of asset pricing or state prices, prove that if the market is free of arbitrage then $\phi^{(1)}\mathbf{S}_0 = \phi^{(2)}\mathbf{S}_0$, where $\mathbf{S}_0$ is the initial price vector.

We argue by contradiction. Suppose that $\phi^{(1)}\mathbf{S}_0 > \phi^{(2)}\mathbf{S}_0$. *Then the portfolio* $\phi = \phi^{(2)} - \phi^{(1)}$ *is an arbitrage. Indeed the cost of this portfolio is*

$$\phi\mathbf{S}_0 = (\phi^{(2)} - \phi^{(1)})\mathbf{S}_0 = \phi^{(2)}\mathbf{S}_0 - \phi^{(1)}\mathbf{S}_0 < 0$$

and its terminal value is

$$\phi D = (\phi^{(2)} - \phi^{(1)})D = \phi^{(2)}D - \phi^{(1)}D = \mathbf{v} - \mathbf{v} = 0\ ,$$

making it a free two-dollar bill. But this contradicts the fact that the market is free of arbitrage, and so it cannot be true that $\phi^{(1)}\mathbf{S}_0 > \phi^{(2)}\mathbf{S}_0$. *Therefore it must be true that* $\phi^{(1)}\mathbf{S}_0 \le \phi^{(2)}\mathbf{S}_0$. *An analogous argument to the one given above proves that it cannot be true that* $\phi^{(1)}\mathbf{S}_0 < \phi^{(2)}\mathbf{S}_0$, *the only remaining possibility is that* $\phi^{(1)}\mathbf{S}_0 = \phi^{(2)}\mathbf{S}_0$.