MA 370 - Summer 2013
Assignment #4 (**Optional**) Solutions

1. In this problem we investigate continuously compounded stock returns in the binomial model. To this end consider the N-period binomial model with parameters u, d and p.

   (a) Determine the mean and standard deviation of the one-period continuously compounded return $\log(S_{n+1}/S_n)$.

   The key here is to recall (July 15 lecture) that

   $$S_{n+1} = S_n u^{Z_{n+1}} d^{1-Z_{n+1}} \ ,$$

   where

   $$Z_{n+1} = \begin{cases} 1 & \text{with probability } p \ , \\ 0 & \text{with probability } 1-p \ . \end{cases}$$

   Observe that Z_{n+1} is a binomial with parameters $n = 1$ and p, so that its mean and variance are p and $p(1-p)$, respectively. Now write the continuously compounded single-period return in terms of Z_{n+1} as follows

   $$\begin{aligned} \log(S_{n+1}/S_n) &= \log(u^{Z_{n+1}} d^{1-Z_{n+1}}) \\ &= \log(u^{Z_{n+1}}) + \log(d^{1-Z_{n+1}}) \\ &= Z_{n+1}\log(u) + (1-Z_{n+1})\log(d) \\ &= \log(d) + [\log(u) - \log(d)]Z_{n+1} \ . \end{aligned}$$

   So the continuously compounded return is a linear function of the random variable Z_{n+1}, which allows us to calculate its mean and variance as follows

   $$\begin{aligned} \mathbb{E}[\log(S_{n+1}/S_n)] &= \mathbb{E}[\log(d) + [\log(u) - \log(d)]Z_{n+1}] \\ &= \log(d) + [\log(u) - \log(d)]\mathbb{E}[Z_{n+1}] \\ &= \log(d) + [\log(u) - \log(d)]p \\ &= p\log(u) + (1-p)\log(d) \ , \end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(\log(S_{n+1}/S_n)) &= \mathrm{Var}(\log(d) + [\log(u) - \log(d)]Z_{n+1}) \\ &= [\log(u) - \log(d)]^2 \mathrm{Var}(Z_{n+1}) \\ &= [\log(u) - \log(d)]^2 p(1-p)\ .\end{aligned}$$

The standard deviation of the single-period return is therefore

$$\mathrm{SD}(\log(S_{n+1}/S_n)) = [\log(u) - \log(d)]\sqrt{p(1-p)}\ .$$

(b) Determine the mean and standard deviation of the m-period continuously compounded return $\log(S_{n+m}/S_n)$.
The trick here is to recall (July 15 again) that

$$S_n = S_0 u^{X_n} d^{n-X_n}\ ,$$

where $X_n = \sum_{k=1}^{n} Z_k$ *and* $Z_1, Z_2, \ldots$ *are independent random variables such that*

$$Z_k = \begin{cases} 1 & \text{with probability } p\ , \\ 0 & \text{with probability } 1-p\ . \end{cases}$$

Recall that we (i) interpreted Z_k *as an indicator random variable, telling us whether the stock moved up or down from period* $k-1$ *to period* k*, and (ii) noted that* X_n *has a binomial distribution with parameters* n *and* p*. Now observe that*

$$\begin{aligned}S_{n+m}/S_n &= S_0 u^{X_{n+m}} d^{(n+m)-X_{n+m}} / S_0 u^{X_n} d^{n-X_n} \\ &= u^{X_{n+m}-X_n} d^{m-(X_{n+m}-X_n)}\ .\end{aligned}$$

Now $X_{n+m} - X_n = \sum_{k=n+1}^{m} Z_k$ *counts the number of times the stock moves up from period* n *to period* $n+m$ *(over which there are* m *total movements) and therefore has a binomial distribution with parameters* m *and* p*. In particular the mean and variance of* $X_{n+m} - X_n$ *are* mp *and* $mp(1-p)$*, respectively. As in (a) the* *trick is to write the continuously compounded m-period return as a linear function of $X_{n+m} - X_n$, as follows*

$$\begin{aligned}\log(S_{n+m}/S_n) &= \log(u^{X_{n+m}-X_n} d^{m-(X_{n+m}-X_n)}) \\ &= (X_{n+m} - X_n)\log(u) + [m - (X_{n+m} - X_n)]\log(d) \\ &= m\log(d) + [\log(u) - \log(d)](X_{n+m} - X_n)\ .\end{aligned}$$

We can now get the mean and variance of the return from that of $X_{n+m} - X_n$ as follows

$$\begin{aligned}\mathbb{E}[\log(S_{n+m}/S_n)] &= \mathbb{E}[m\log(d) + [\log(u) - \log(d)](X_{n+m} - X_n)] \\ &= m\log(d) + [\log(u) - \log(d)]\mathbb{E}[X_{n+m} - X_n] \\ &= m\log(d) + [\log(u) - \log(d)]mp \\ &= mp\log(u) + m(1-p)\log(d) \\ &= m[p\log(u) + (1-p)\log(d)]\ ,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(\log(S_{n+m}/S_n)) &= \mathrm{Var}(m\log(d) + [\log(u) - \log(d)](X_{n+m} - X_n)) \\ &= [\log(u) - \log(d)]^2\mathrm{Var}(X_{n+m} - X_n) \\ &= [\log(u) - \log(d)]^2 mp(1-p)\ .\end{aligned}$$

The standard deviation of the m-period return is therefore

$$\mathrm{SD}(\log(S_{n+m}/S_n)) = [\log(u) - \log(d)]\sqrt{mp(1-p)}\ .$$

It is interesting to note that

$$\begin{aligned}\mathbb{E}[\log(S_{n+m}/S_n)] &= m \cdot \mathbb{E}[\log(S_{n+1}/S_n)]\ , \\ \mathrm{SD}(\log(S_{n+m}/S_n)) &= \sqrt{m} \cdot \mathrm{SD}[\log(S_{n+1}/S_n)]\ .\end{aligned}$$

So the expected return over $m = 10$ days is just ten times the expected daily return, while the standard deviation of the ten-day return is $\sqrt{10} \approx 3.16$ times the daily standard deviation. Or, the *annual return is 250 times the daily return, while the annual standard deviation is only* $\sqrt{250} \approx 15.8$ *times the daily standard deviation. This becomes important when one uses daily data (for which there is lots of data/information) in order to make inferences on annual parameters (for which there is less data/information)*

2. In this problem we indicate how one might build a binomial tree in practice, i.e. how one might actually choose the parameters of the tree in order to match statistical properties of the underlying stock. To this end suppose that I am interested in building a binomial tree to price and hedge an option with a maturity of 2 years. Further suppose that I would like my tree to have 250 nodes per year, so that each node corresponds to one trading day. Looking at historical data I observe that on average, the continuously compounded annual return on this stock is 14% with a volatility (standard deviation) of 45%. Furthermore, I find that the stock generates a positive return on 60% of the days in my historical sample. Show that if I set $p = 0.6$,

$$u = \exp\left(\frac{0.14}{250} + \sqrt{\frac{0.4}{0.6}}\sqrt{\frac{0.45}{250}}\right) \ , \quad d = \exp\left(\frac{0.14}{250} - \sqrt{\frac{0.6}{0.4}}\sqrt{\frac{0.45}{250}}\right) \ ,$$

then I have built a sensible tree. More precisely show that in a binomial model with these parameters

(a) The probability that $\log(S_{n+1}/S_n)$ is positive is 0.6.
In part (a) of the previous problem we saw that the daily return can be written

$$\log(S_{n+1}/S_n) = \log(d) + [\log(u) - \log(d)]Z_{n+1} \ ,$$

which means that

$$\log(S_{n+1}/S_n) = \begin{cases} \log(u) & \text{with probability } p \ , \\ \log(d) & \text{with probability } 1 - p \ . \end{cases}$$

Plugging in the given values for p, u and d we get

$$\log(S_{n+1}/S_n) = \begin{cases} \frac{0.14}{250} + \sqrt{\frac{0.4}{0.6}}\sqrt{\frac{0.45}{250}} \approx 0.035 & \text{with probability } 0.6\ , \\ \frac{0.14}{250} - \sqrt{\frac{0.6}{0.4}}\sqrt{\frac{0.45}{250}} \approx -0.051 & \text{with probability } 0.4\ . \end{cases}$$

Hence the probability that the daily return is positive is 60%, as expected. It is worth noting that, using these parameters, the stock will either rise by 3.5% (which happens 60% of the time) or fall by 5.1% (which happens 40% of the time) on any given day.

(b) The mean of $\log(S_{n+250}/S_n)$ is 0.14.
Using the formula from Problem 1(b) we get

$$\begin{aligned}
\mathbb{E}[\log(S_{n+250}/S_n)] &= 250[(.6)\log(u) + (.4)\log(d)] \\
&= 250\left[(.6)\left(\frac{0.14}{250} + \sqrt{\frac{0.4}{0.6}}\sqrt{\frac{0.45}{250}}\right)\right. \\
&\qquad \left. +(.4)\left(\frac{0.14}{250} - \sqrt{\frac{0.6}{0.4}}\sqrt{\frac{0.45}{250}}\right)\right] \\
&= 250\left[\frac{(0.6)(0.14) + (0.4)(0.14)}{250}\right. \\
&\qquad \left. +\sqrt{(0.4)(0.6)}\sqrt{\frac{0.45}{250}} - \sqrt{(0.6)(0.4)}\sqrt{\frac{0.45}{250}}\right] \\
&= 250 \cdot \frac{0.14}{250} \\
&= 0.14\ ,
\end{aligned}$$

as expected.

(c) The volatility (standard deviation) of $\log(S_{n+250}/S_n)$ is 0.45.

Using the formula from Problem 1(b) we get

$$\begin{aligned}
\mathrm{SD}(\log(S_{n+250}/S_n)) &= [\log(u) - \log(d)]\sqrt{250(0.6)(0.4)} \\
&= \sqrt{250(0.6)(0.4)}\left[\frac{0.14}{250} + \sqrt{\frac{0.4}{0.6}}\sqrt{\frac{0.45}{250}} \right. \\
&\qquad \left. -\frac{0.14}{250} + \sqrt{\frac{0.6}{0.4}}\sqrt{\frac{0.45}{250}}\right] \\
&= \sqrt{250(0.6)(0.4)} \cdot \sqrt{\frac{0.45}{250}}\left[\sqrt{\frac{0.4}{0.6}} + \sqrt{\frac{0.6}{0.4}}\right] \\
&= \sqrt{0.45}\sqrt{(0.6)(0.4)}\left[\sqrt{\frac{0.4}{0.6}} + \sqrt{\frac{0.6}{0.4}}\right] \\
&= \sqrt{0.45}\,[0.4 + 0.6] \\
&= \sqrt{0.45}\ ,
\end{aligned}$$

as expected.

3. Consider a three-period binomial model with parameters $u = 1.23$ and $d = 0.87$ (the value of p will not influence the answer to the problem, so we don't need to worry about it). Assume the stock is currently trading at $S_0 = 50$ and the risk-free rate of interest is $r = 0.05$.

   (a) Determine the fair value of a Europen put option struck at $K = 50$ by using the recursive procedure discussed in lecture.
   A tree for the stock price is as follows

			93.04
		75.65	
	61.50		65.81
50		53.51	
	43.50		46.55
		37.85	
			32.93

To determine the value of the put option we start with the terminal nodes, where the value at node $(i,3)$ *is* $p(i,3) = \max(50 - S(i,3), 0)$*, leading to*

$$
\begin{array}{cccc}
 & & & 0.00 \\
 & & p(2,2) & \\
 & p(1,1) & & 0.00 \\
p(0,0) & & p(1,2) & \\
 & p(0,1) & & 3.45 \\
 & & p(0,2) & \\
 & & & 17.07
\end{array}
$$

To fill in earlier nodes we use the recursive formula from lecture

$$p(i,n) = q \cdot \frac{p(i+1, n+1)}{1+r} + (1-q) \cdot \frac{p(i, n+1)}{1+r} \ ,$$

for example (note that $q = \frac{(1.05)-0.87}{1.23-0.87} = \frac{0.18}{0.36} = 0.5$*)*

$$
\begin{aligned}
p(1,2) &= q \cdot \frac{p(2,3)}{1+r} + (1-q) \cdot \frac{p(1,3)}{1+r} \\
&= (0.5) \cdot \frac{0.00}{1.05} + (0.5) \cdot \frac{3.45}{1.05} = 1.64 \ .
\end{aligned}
$$

Observe that $p(1,2)$ *is the value of the European put at node* $(1,2)$*, and is not the same as the intrinsic value of the option at that node (which is* $\max(50 - S(1,2), 0) = 0.00$*). Following this procedure until we get to the initial node we end up with*

$$
\begin{array}{cccc}
 & & & 0.00 \\
 & & 0.00 & \\
 & 0.78 & & 0.00 \\
2.96 & & 1.64 & \\
 & 5.44 & & 3.45 \\
 & & 9.77 & \\
 & & & 17.07
\end{array}
$$

So the current/initial value of the put is \$2.96.

(b) Determine the fair value of a Europen put option struck at $K = 50$ by calculating the expected present value of its payoff in the risk-neutral world.

The payoff at node $(i, 3)$ *is* $\max(K - S(i,3), 0)$ *and, in the risk-neutral world, the probability that we end up at node* $(i, 3)$ *is* $\binom{3}{i} q^i (1-q)^{3-i}$*. The desired expectation is therefore*

$$\begin{aligned}
\mathbb{E}_q\left[\frac{\max(K - S_3, 0)}{(1+r)^3}\right] &= \frac{\max(50 - 93.04, 0)}{(1.05)^3} \cdot \binom{3}{3} q^3 (1-q)^0 \\
&\quad + \frac{\max(50 - 65.81, 0)}{(1.05)^3} \cdot \binom{3}{2} q^2 (1-q)^1 \\
&\quad + \frac{\max(50 - 46.55, 0)}{(1.05)^3} \cdot \binom{3}{1} q^1 (1-q)^2 \\
&\quad + \frac{\max(50 - 32.93, 0)}{(1.05)^3} \cdot \binom{3}{3} q^0 (1-q)^3 \\
&= 0 + 0 + \frac{3.45}{(1.05)^3} \cdot 3(.5)^3 + \frac{17.07}{(1.05)^3} \cdot (.5)^3 \\
&= 2.96\ .
\end{aligned}$$

(c) Sketch a “delta tree” that tells me exactly how many shares I need to be short at every node in order to hedge a short position in a call option struck at $K = 50$.

There is a typo in the problem, we are hedging a put and not a call. The formula here is, for $0 \le n \le 2$ *and* $0 \le i \le n$*,*

$$\Delta(i, n) = \frac{p(i+1, n+1) - p(i, n+1)}{S(i+1, n+1) - S(i, n+1)}\ .$$

For example if I were to find myself short one put option at node $(0, 1)$ *then, going forward, I had better be short*

$$\Delta(0, 1) = \frac{p(1,2) - p(0,2)}{S(1,2) - S(0,2)} = \frac{1.64 - 9.77}{53.51 - 37.85} = \frac{-8.13}{15.66} = -0.5192$$

shares in order to make sure my short position is hedged over the next period.

			N/A
		0	
	-0.0742		N/A
-0.2586		-0.1791	
	-0.5192		N/A
		-1	
			N/A

(d) Suppose I sell the option (today) for its fair value and then set up a dynamic hedge. Describe how the dynamic hedge would unfold (i.e. describe the action taken at each period, as well as the associated cash flows) if the stock were to follow the trajectory $50 \mapsto 61.50 \mapsto 53.51 \mapsto 46.55$.

Initially I sell the put for \$2.96 and short 0.2586 *shares. The short brings in an extra (on top of the 2.96 from selling the put)* $(.2586)50 = 12.93$*, and so in total I have \$15.89, which I'll deposit at 5% per period.*

- *If the stock goes up to* 61.50 *then we find ourselves at node* $(1,1)$*, and our delta tree tells us we need to be short* 0.0742 *shares, going forward, in order to maintain our hedge. This means we are too short, and we need to purchase* $0.2586 - 0.0742 = 0.1844$ *shares. This will cost us* $(.1844)61.50 = 11.34$*, which we will take out of the bank. This leaves us with a balance of* $15.89(1.05) - 11.34 = 5.34$*. Going forward, therefore, we are short* 0.0742 *shares and have a balance in our chequing account of* 5.34*. Observe that our net position is* $5.34 - (.0742)61.50 = 0.78$*, which is exactly the value of the put option at node* $(1,1)$*.*
- *If the stock now drops to* 53.51 *then we find ourselves at node* $(1,2)$ *and our delta tree indicates we need to be short* 0.1791 *shares. So we'll have to short an additional* $0.1791 - 0.0742 = 0.1049$ *shares, bringing in* $(.1049)53.51 = 5.61$*. We put this*

in the bank, taking our balance to $5.34(1.05) + 5.61 = 11.22$. *So going forward we're short* 0.1791 *shares and have a bank balance of* 11.22. *Observe that our net position is* $11.22 - (.1791)53.51 = 1.64$, *which is precisely the value of the option that we're hedging at node* $(1,2)$.

- *Now the stock drops to* 46.55, *putting us at node* $(1,3)$. *We owe our counterparty* 3.45 *and have to pay* $(.1791)46.55 = 8.34$ *to close our short position in the stock. This costs us* $3.45+8.34 = 11.79$*; lo and behold we have* $11.22(1.05) = 11.78$ *in the bank (had we not been rounding along the way these two numbers would have matched). Modulo rounding error we simply empty our bank account and use the proceeds to pay off our counterparty and close our short position in the stock, leaving us with zero profit/loss, as expected.*

(e) Determine the fair value of a European derivative that pays one dollar in the event that the terminal stock price is somewhere between \$40 and \$70 (buying such a derivative is effectively a bet that the return on the stock will not be too extreme), and sketch its corresponding delta tree.

Let $f(i,n)$ *denote the fair value of the derivative at node* (i,n). *Then the value at any terminal node is*

$$f(i,3) = \begin{cases} 1 & \text{if} \quad 40 \leq S(i,3) \leq 70 \ , \\ 0 & \text{if} \quad S(i,3) < 40 \text{ or } S(i,3) > 70 \ . \end{cases}$$

Thus we have

			0.00
		$f(2,2)$	
	$f(1,1)$		1.00
$f(0,0)$		$f(1,2)$	
	$f(0,1)$		1.00
		$f(0,2)$	
			0.00

The easiest way to get the fair value from here is to calculate the expected present value, in the risk-neutral world, of its terminal payoff

$$f_0 = 0 + \frac{1}{(1.05)^3} \cdot 3(.5)^3 + \frac{1}{(1.05)^3} \cdot 3(.5)^3 + 0 = 0.65 \ .$$

In order to fill in the delta tree we'll need to determine the value at all nodes, which can be accomplished via the recursion

$$f(i,n) = q \cdot \frac{f(i+1,n+1)}{1+r} + (1-q) \cdot \frac{f(i,n+1)}{1+r} \ ,$$

for example

$$\begin{aligned} f(0,2) &= q \cdot \frac{f(1,3)}{1+r} + (1-q) \cdot \frac{f(0,3)}{1+r} \\ &= (0.5) \cdot \frac{1.00}{1.05} + (0.5) \cdot \frac{0.00}{1.05} = 0.48 \ . \end{aligned}$$

Following this procedure until we get to the initial node we end up with

			0.00
		0.48	
	0.68		1.00
0.65		0.95	
	0.68		1.00
		0.48	
			0.00

Using the recursive procedure to calculate the derivative's delta we get

$$
\begin{array}{cccc}
 & & & N/A \\
 & & -0.0367 & \\
 & -0.0215 & & N/A \\
0 & & 0 & \\
 & 0.0304 & & N/A \\
 & & 0.0734 & \\
 & & & N/A
\end{array}
$$

4. Consider a five-period binomial model with parameters $u = 1.17$ and $d = 0.89$ (the value of p will not influence the answer to the problem, so we don't need to worry about it). Suppose I sell a call option struck at $K = 50$ on a stock that is currently trading at $S_0 = 50$ and put a dynamic hedge in place. The risk-free rate of interest is 4%. If the stock is trading at \$60.92 after three periods

   (a) How many shares do I own immediately after my period-three rebalance?

   Given that the stock is currently trading at 60.92 *we are effectively in a two-period world whose stock/option/delta tree is*

$$
\begin{array}{ccc}
 & & 83.39/33.39/NA \\
 & 71.28/23.20/1 & \\
60.92/15.04/0.9543 & & 63.44/13.44/NA \\
 & 54.22/6.92/0.8850 & \\
 & & 48.25/0/NA
\end{array}
$$

   So immediately after my period-three rebalance I own 0.9543 *shares.*

   (b) How many bonds am I short immediately after my period-three rebalance?

   Using the notation from lecture on July 22 we have that $\Delta_n S_n + \beta_n B_n = c_n$*, where* c_n *is the value of the call option at period* n*. Here we are interested in* β_3*, the number of bonds we are short* *after the period-three rebalance, which is given by*

$$\begin{aligned}\beta_3 &= \frac{c_3 - \Delta_3 S_3}{B_3} \\ &= \frac{15.04 - (0.9543)(60.92)}{(1.04)^{-2}} \\ &= \frac{-43.0960}{0.9246} \\ &= -46.6104\ .\end{aligned}$$

Therefore I am short 46.6104 *bonds.*

(c) What is my total outstanding debt immediately after my period-three rebalance?
My total outstanding debt is

$$\begin{aligned}-\beta_3 B_3 &= c_3 - \Delta_3 S_3 \\ &= -43.0960\ .\end{aligned}$$

So I owe 43.0960 *in total. Since I own* 0.9543 *shares my net position is* $(0.9543)(60.92) - 43.0960 = 15.04$, *which is exactly the value of the derivative.*

5. In a one-period binomial model prove that if $S_u < K$ then

(a) $S_0 < \frac{K}{1+r}$.
If $S_u < K$ *then* $S_d < K$ *as well and we have*

$$\begin{aligned}S_0 &= q \cdot \frac{S_u}{1+r} + (1-q) \cdot \frac{S_d}{1+r} \\ &< q \cdot \frac{K}{1+r} + (1-q) \cdot \frac{K}{1+r} \\ &= \frac{K}{1+r}\ .\end{aligned}$$

(b) The fair value of a European put option struck at K is $p_0 = \frac{K}{1+r} - S_0$.

Since $S_d < K$ *and* $S_u < K$ *we have* $\max(K - S_d, 0) = K - S_d$ *and* $\max(K - S_u, 0) = K - S_u$. *Thus*

$$\begin{aligned} p_0 &= q \cdot \frac{\max(K - S_u, 0)}{1+r} + (1-q) \cdot \frac{\max(K - S_d, 0)}{1+r} \\ &= q \cdot \frac{K - S_u}{1+r} + (1-q) \cdot \frac{K - S_d}{1+r} \\ &= \frac{q(K - S_u) + (1-q)(K - S_d)}{1+r} \\ &= \frac{qK + (1-q)K - [qS_u + (1-q)S_d]}{1+r} \\ &= \frac{K - [qS_u + (1-q)S_d]}{1+r} \\ &= \frac{K}{1+r} - \left[q \cdot \frac{S_u}{1+r} + (1-q) \cdot \frac{S_d}{1+r} \right] \\ &= \frac{K}{1+r} - S_0 \, . \end{aligned}$$

6. Consider a three-period binomial model with $u = 1.32$ and $d = 1/u$. Assume the risk-free rate of interest is $r = 0.05$ and consider an American put option struck at $K = 60$ on a stock that is currently trading at $S_0 = 50$.

   (a) Fill in a tree that illustrates the (i) intrinsic value, (ii) continuation value and (iii) overall option value at every node. Be sure to indicate all nodes at which early exercise is optimal.
   The stock/intrinsic/continuation tree is as follows. Bold indicates overall option value and nodes where the option should be exercised are indicated with a star.

			115.00/0.00/**0.00**
		87.12/0.00/**0.00**	
	66.00/0.00/**4.62**		66.00/0.00/**0.00**
50.00/10.00/**12.40**		50.00/10.00/**10.11**	
	∗37.88/**22.12**/19.32		∗37.88/**22.12**/0.00
		∗28.70/**31.30**/28.45	
			∗21.74/**38.26**/0.00

(b) How much more valuable is the American option than its European counterpart?

The value of the corresponding European is 10.53*, so the American option is worth* 1.87 *more than its European counterpart.*

The following two problems will not be marked and do not need to be handed in, but you should consider them practice problems for the final exam.

7. Recall that a self-financing trading strategy is a pair of stochastic processes (δ_n, α_n) that (i) is adapted to the market filtration $\mathcal{F}_n = \sigma(S_0, S_1, S_2, \ldots, S_n)$ and (ii) satisfies the self-financing condition $\delta_n S_n + \alpha_n B_n = \delta_{n-1} S_n + \alpha_{n-1} B_n$. Let $V_n = \delta_n S_n + \alpha_n B_n$ denote the value of such a trading strategy immediately after reblancing at time n.

   (a) Show that $E_q[V_{n+1}|\mathcal{F}_n] = (1+r)V_n$ for any self-financing trading strategy, where E_q denotes expectation in the risk-neutral world (i.e. expected value calculated using the risk-neutral probability).

   Using the self-financing property and linearity of conditional expectations we have

$$\begin{aligned} E_q[V_{n+1}|\mathcal{F}_n] &= E_q[\delta_{n+1} S_{n+1} + \alpha_{n+1} B_{n+1}|\mathcal{F}_n] \\ &= E_q[\delta_n S_{n+1} + \alpha_n B_{n+1}|\mathcal{F}_n] \\ &= E_q[\delta_n S_{n+1}|\mathcal{F}_n] + E_q[\alpha_n B_{n+1}|\mathcal{F}_n] \\ &= \delta_n E_q[S_{n+1}|\mathcal{F}_n] + \alpha_n E_q[B_{n+1}|\mathcal{F}_n] \ , \end{aligned}$$

since each of δ_n and α_n is measurable with respect to $\mathcal{F}_n$. Now B_{n+1} is not actually random, and so $E_q[B_{n+1}|\mathcal{F}_n] = B_{n+1} = (1+r)B_n$, and in the final lecture we showed that $E_q[S_{n+1}|\mathcal{F}_n] = (1+r)S_n$. Thus we have

$$\begin{aligned} E_q[V_{n+1}|\mathcal{F}_n] &= \delta_n(1+r)S_n + \alpha_n(1+r)B_n \\ &= (1+r)[\delta_n S_n + \alpha_n B_n] \\ &= (1+r)V_n\ , \end{aligned}$$

as required.

(b) Use (a) to show that if we define $\overline{V}_n = (1+r)^{-n}\overline{V}_n$ as the discounted value of the self-financing strategy, then $\overline{V}_n$ is a martingale in the risk-neutral world.

There is an obvious typo here, we should have $\overline{V}_n = (1+r)^{-n}V_n$. To show that this process is a martingale in the risk-neutral world we must show that $E_q[\overline{V}_{n+1}|\mathcal{F}_n] = \overline{V}_n$. To this end we have (in going from line two to three we use the result from (a))

$$\begin{aligned} E_q[\overline{V}_{n+1}|\mathcal{F}_n] &= E_q[V_{n+1}/(1+r)^{n+1}|\mathcal{F}_n] \\ &= (1+r)^{-(n+1)}E_q[V_{n+1}|\mathcal{F}_n] \\ &= (1+r)^{-(n+1)}(1+r)V_n \\ &= (1+r)^{-n}V_n \\ &= \overline{V}_n\ , \end{aligned}$$

as required.

8. Consider a three-period binomial tree with $u = 1.1$ and $d = 1/u$. Suppose we are interested in hedging a European call option struck at $K = 100$ on a stock that is currently trading at $S_0 = 100$ and that the risk-free rate of interest is 4%.

(a) Show that $S(0,0) = S(1,2)$ but $\Delta(0,0) \neq \Delta(1,2)$.

The stock/option/delta tree is

			133.10/33.10/NA
		121/24.85/1.0000	
	110.00/18.37/0.8692		110.00/10.00/NA
100.00/13.43/0.7348		100.00/6.59/0.5238	
	90.91/4.35/0.3799		90.91/0.000/NA
		82.64/0.00/0.000	
			75.13/0.00/NA

The stock price at nodes $(0,0)$ *and* $(1,2)$ *is* 100, *but* $\Delta(0,0) = 0.73$ *while* $\Delta(1,2) = 0.52 < \Delta(0,0)$.

(b) If the stock price is the same at these two nodes, why isn't the hedge ratio the same? In particular, why is $\Delta(0,0)$ larger than $\Delta(1,2)$?

The short position in the call at node $(0,0)$ *is far more dangerous than the short position at* $(1,2)$, *since there is more time for the option to get deep into the money. By contrast the damage is, comparatively speaking, limited at node* $(1,2)$. *Therefore we need more insurance, i.e. a larger hedge, at the "more dangerous node."*