MA 370 - Summer 2013

Assignment #4 (**Optional**)

- Due (if you choose to hand it in) Friday, August 2 by 4:00 p.m.
- You may hand in your assignment at any time prior to the due date, either in my office (MC 268 - slip it under the door if I am not there) or in class.
- There are eight problems, but only the first six will be marked. The last two are included as extra practice for the final exam.

1. In this problem we investigate continuously compounded stock returns in the binomial model. To this end consider the N-period binomial model with parameters u, d and p.

   (a) Determine the mean and standard deviation of the one-period continuously compounded return $\log(S_{n+1}/S_n)$.

   (b) Determine the mean and standard deviation of the m-period continuously compounded return $\log(S_{n+m}/S_n)$.

   In both cases you may use the fact that, if Z has a binomial distribution with parameters n and p then the mean and variance of Z are np and $np(1-p)$, respectively.

2. In this problem we indicate how one might build a binomial tree in practice, i.e. how one might actually choose the parameters of the tree in order to match statistical properties of the underlying stock. To this end suppose that I am interested in building a binomial tree to price and hedge an option with a maturity of 2 years. Further suppose that I would like my tree to have 250 nodes per year, so that

each node corresponds to one trading day. Looking at historical data I observe that on average, the continuously compounded annual return on this stock is 14% with a volatility (standard deviation) of 45%. Furthermore, I find that the stock generates a positive return on 60% of the days in my historical sample. Show that if I set $p = 0.6$,

$$u = \exp\left(\frac{0.14}{250} + \sqrt{\frac{0.4}{0.6}}\sqrt{\frac{0.45}{250}}\right), \quad d = \exp\left(\frac{0.14}{250} - \sqrt{\frac{0.6}{0.4}}\sqrt{\frac{0.45}{250}}\right),$$

then I have built a sensible tree. More precisely show that in a binomial model with these parameters

(a) The probability that $\log(S_{n+1}/S_n)$ is positive is 0.6.

(b) The mean of $\log(S_{n+250}/S_n)$ is 0.14.

(c) The volatility (standard deviation) of $\log(S_{n+250}/S_n)$ is 0.45.

3. Consider a three-period binomial model with parameters $u = 1.23$ and $d = 0.87$ (the value of p will not influence the answer to the problem, so we don't need to worry about it). Assume the stock is currently trading at $S_0 = 50$ and the risk-free rate of interest is $r = 0.05$.

   (a) Determine the fair value of a Europen put option struck at $K = 50$ by using the recursive procedure discussed in lecture.

   (b) Determine the fair value of a Europen put option struck at $K = 50$ by calculating the expected present value of its payoff in the risk-neutral world.

   (c) Sketch a "delta tree" that tells me exactly how many shares I need to be short at every node in order to hedge a short position in a call option struck at $K = 50$.

   (d) Suppose I sell the option (today) for its fair value and then set up a dynamic hedge. Describe how the dynamic hedge would unfold (i.e. describe the action taken at each period, as well as the associated cash flows) if the stock were to follow the trajectory $50 \mapsto 61.50 \mapsto 53.51 \mapsto 46.55$.

(e) Determine the fair value of a European derivative that pays one dollar in the event that the terminal stock price is somewhere between \$40 and \$70 (buying such a derivative is effectively a bet that the return on the stock will not be too extreme), and sketch its corresponding delta tree.

4. Consider a five-period binomial model with parameters $u = 1.17$ and $d = 0.89$ (the value of p will not influence the answer to the problem, so we don't need to worry about it). Suppose I sell a call option struck at $K = 50$ on a stock that is currently trading at $S_0 = 50$ and put a dynamic hedge in place. The risk-free rate of interest is 4%. If the stock is trading at \$60.92 after three periods

   (a) How many shares do I own immediately after my period-three rebalance?

   (b) How many bonds am I short immediately after my period-three rebalance?

   (c) What is my total outstanding debt immediately after my period-three rebalance?

5. In a one-period binomial model prove that if $S_u < K$ then

   (a) $S_0 < \frac{K}{1+r}$.

   (b) The fair value of a European put option struck at K is $p_0 = \frac{K}{1+r} - S_0$.

6. Consider a three-period binomial model with $u = 1.32$ and $d = 1/u$. Assume the risk-free rate of interest is $r = 0.05$ and consider an American put option struck at $K = 60$ on a stock that is currently trading at $S_0 = 50$.

(a) Fill in a tree that illustrates the (i) intrinsic value, (ii) continuation value and (iii) overall option value at every node. Be sure to indicate all nodes at which early exercise is optimal.

(b) How much more valuable is the American option than its European counterpart?

The following two problems will not be marked and do not need to be handed in, but you should consider them practice problems for the final exam.

7. Recall that a self-financing trading strategy is a pair of stochastic processes (δ_n, α_n) that (i) is adapted to the market filtration $\mathcal{F}_n = \sigma(S_0, S_1, S_2, \ldots, S_n)$ and (ii) satisfies the self-financing condition $\delta_n S_n + \alpha_n B_n = \delta_{n-1} S_n + \alpha_{n-1} B_n$. Let $V_n = \delta_n S_n + \alpha_n B_n$ denote the value of such a trading strategy immediately after reblancing at time n.

   (a) Show that $E_q[V_{n+1}|\mathcal{F}_n] = (1+r)V_n$ for any self-financing trading strategy, where E_q denotes expectation in the risk-neutral world (i.e. expected value calculated using the risk-neutral probability).

   (b) Use (a) to show that if we define $\overline{V}_n = (1+r)^{-n}\overline{V}_n$ as the discounted value of the self-financing strategy, then $\overline{V}_n$ is a martingale in the risk-neutral world.

8. Consider a three-period binomial tree with $u = 1.1$ and $d = 1/u$. Suppose we are interested in hedging a European call option struck at $K = 100$ on a stock that is currently trading at $S_0 = 100$ and that the risk-free rate of interest is 4%.

   (a) Show that $S(0,0) = S(1,2)$ but $\Delta(0,0) \neq \Delta(1,2)$.

   (b) If the stock price is the same at these two nodes, why isn't the hedge ratio the same? In particular, why is $\Delta(0,0)$ larger than $\Delta(1,2)$?