MA370 - Summer 2013
Assignment #3 Solutions

1. Let $\Omega = \{1, 2, 3, 4\}$. Which (if any) of the following collections are fields/algebras?

   (a) $\mathcal{F}_1 = \{\emptyset, \{1, 2\}, \{3, 4\}\}$.
   Not a field since $\Omega \notin \mathcal{F}_1$.

   (b) $\mathcal{F}_2 = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}\}$.
   Not a field since $\{1\} \in \mathcal{F}_2$ *and* $\{3, 4\} \in \mathcal{F}_2$ *but* $\{1\} \cup \{3, 4\} = \{1, 3, 4\} \notin \mathcal{F}_2$.

   (c) $\mathcal{F}_3 = \{\emptyset, \Omega, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\}$.
   Field, to check simply confirm that if A *and* B *are any sets in* $\mathcal{F}_3$ *then* $A^C \in \mathcal{F}_3$ *and* $A \cup B \in \mathcal{F}_3$.

2. Let $\Omega = [0, 1]$. Adding as few sets as possible, complete the family of sets $\{\emptyset, [0, 1/2), \{1\}\}$ to obtain a field on Ω.
   Here we're looking for the field generated by the sets $A = [0, 1/2)$ *and* $B = \{1\}$. *We gave a general "formula" for this in lecture on June 19, specializing it to the current pair A,B we see that the field of interest*

will have to contain

$$\begin{aligned}
A^C &= [1/2, 1] \\
B^C &= [0, 1) \\
A \cup B &= [0, 1/2) \cup \{1\} \\
A \cup B^C &= [0, 1) \\
A^C \cup B &= [1/2, 1] \\
A^C \cup B^C &= [0, 1] \\
A \cap B &= \emptyset \\
A^C \cap B &= \{1\} \\
A \cap B^C &= [0, 1/2) \\
A^C \cap B^C &= [1/2, 1)\ .
\end{aligned}$$

So the desired field is

$$\{\emptyset, [0,1], [0,1/2), \{1\}, [1/2,1], [0,1), [0,1/2) \cup \{1\}, [1/2,1)\}\ .$$

3. Suppose I play three rounds of a game where, on each round, I either win or lose one dollar. Let Z_k denote my net winnings on round k for $k = 1, 2, 3$, so that Z_k is a random variable that either takes on the value 1 (if I win round k) or -1 (if I lose round k). Let $X_0 = 0$ and $X_n = \sum_{k=1}^{n} Z_k$ for $n = 1, 2, 3$, so that X_n denotes my total winnings after n rounds. Finally, let $M_n = \max(X_0, X_1, \ldots, X_n)$ for $n = 0, 1, 2, 3$. *Before we start note that the sample space here is*

$$\Omega = \{WWW, WWL, WLW, WLL, LWW, LWL, LLW, LLL\}\ .$$

It is useful to sketch a graph of my cumulative winnings through time here (as we did in lecture on July 10) in order to understand what is going on.

(a) Is X_3 is measurable with respect to $\sigma(M_3)$? If it is, explain how the value of X_3 can be deduced from that of M_3. If it is not, give an example of an event A such that $A \in \sigma(X_3)$ but $A \notin \sigma(M_3)$.

X_3 is not measurable with respect to $\sigma(M_3)$. Intuitively this is because the value of X_3 can not necessarily be determined from that of M_3; in other words X_3 is not a function of M_3. For instance there are three outcomes that lead to $M_3 = 1$ and they are WLW, LWW and WLL. In the first two cases we have $X_3 = 1$ but in the third case we have $X_3 = -1$. So if you only tell me that the maximum was 1, I can determine that X_3 is either $+1$ or -1 but nothing else. Learning that $M_3 = 1$ gives me a bit of information on X_3 (for example I now know that X_3 is not $+3$ or -3) but doesn't tell me the whole story. Note also that if you told me $M_3 = 3$ then I would know for sure that $X_3 = 3$, but measurability requires you to be able to ascertain the value of X_3 for all possible values of M_3.

More formally to prove that X_3 is not measurable with respect to $\sigma(M_3)$ we must show that $\sigma(X_3)$ is not a subset of $\sigma(M_3)$, and this requires us to find an element of the former that is not also an element of the latter. To this end observe that $\sigma(M_3)$ is the algebra generated by the partition

$$\begin{aligned}
B_0 &= \{M_3 = 0\} = \{LWL, LLW, LLL\} , \\
B_1 &= \{M_3 = 1\} = \{WLW, LWW, WLL\} , \\
B_2 &= \{M_3 = 2\} = \{WWL\} \\
B_3 &= \{M_3 = 3\} = \{WWW\} ,
\end{aligned}$$

hence it consists of all possible unions constructed using the above sets.

The event $\{X_3 = 1\} = \{WWL, WLW, LWW\}$ lies in $\sigma(X_3)$ but is clearly not a union of any combination of the above sets. If it were possible to write $\{X_3 = 1\}$ as a union of the B_i then that *union would have to involve B_1 since we need both WLW and LWW, both of which lie in B_1. But if we include B_1 in the union then we're going to end up with WLL which is not an element of $\{X_3 = 1\}$.*

In summary, the event $\{X_3 = 1\}$ is not in $\sigma(M_3)$ which means that X_3 is not measurable with respect to $\sigma(M_3)$. Intuitively this means that, based only on the value of M_3 it is never possible to ascertain that $X_3 = 1$ (though in some cases, for example $M_3 = 3$, it is possible to ascertain that $X_3 \neq 1$).

(b) Is Z_2 measurable with respect to $\sigma(X_2)$? If it is, explain how the value of Z_2 can be deduced from that of X_2. If it is not, give an example of an event A such that $A \in \sigma(Z_2)$ but $A \notin \sigma(X_2)$.

No Z_2 is not measurable with respect to $\sigma(X_2)$. For instance if $X_2 = 0$ then Z_2 could either be $+1$ or -1 (note that if $X_2 = 2$ then we know that $Z_2 = 2$). So it is not necessarily possible to determine the value of Z_2 from that of X_2, and therefore we do not have measurability.

More formally, $\sigma(X_2)$ is generated by the partition

$$\begin{aligned} A_1 &= \{X_2 = -2\} = \{LLW, LLL\}\,, \\ A_2 &= \{X_2 = 0\} = \{WLW, WLL, LWW, LWL\}\,, \\ A_3 &= \{X_2 = 2\} = \{WWW, WWL\} \end{aligned}$$

and $\{Z_2 = 1\} = \{WWW, WWL, LWW, LWL\}$ is clearly not a union of the A_i. If it were that union would have to involve A_2 (in order to get LWW and LWL) but that would bring along unwanted elements such as WLL.

(c) Is Z_2 measurable with respect to $\sigma(X_1, X_2)$? If it is, explain how the value of Z_2 can be deduced from those of X_1 and X_2. If it is not, give an example of an event A such that $A \in \sigma(Z_2)$ but $A \notin \sigma(X_1, X_2)$.

Yes it is measurable, since $Z_2 = X_2 - X_1$ by definition. Hence one can always determine the value of Z_2 based on those of X_1, X_2 by using this formula.

4. Two fair dice are tossed, leading to a sample space

$$\Omega = \{(1,1), (1,2), \ldots, (1,6), (2,1), \ldots, (6,6)\} ,$$

where (i,j) indicates that i dots were showing on the first die and j dots were showing on the second die. Let X_k denote the number of dots showing on die k for $k = 1, 2$, and let $Y = X_1 + X_2$. Describe the information set corresponding to each of the following fields (i.e. answer the question "if $\mathcal{F}$ is your information set, then what do you know?"), and in each case determine which of the variables X_1, X_2, Y are measurable with respect to the given field.

(a) $\mathcal{F} = \sigma(B_1, \ldots, B_6)$, where $B_i = \{(i,1), (i,2), \ldots, (i,6)\}$.
B_i is the event that the first die shows i. Thus if $\mathcal{F}$ is your information set you know exactly what happened with the first die. X_1 is therefore clearly measurable with respect to $\mathcal{F}$, but X_2 and Y are not. For example if $X_1 = 6$ then X_2 could be anything and Y could either be $7, 8, 9, 10, 11$ or 12. Notice that X_1 tells us absolutely nothing about X_2 since these variables are independent, but it does tell us a bit about Y (originally we though Y could be anything between 2 and 12, but if $X_1 = 6$ we know it actually has to be between 7 and 12) since X_1 and Y are not independent.

(b) $\mathcal{F} = \sigma(A)$, where

$$A = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$$

A is the event that we got a (strictly) larger number on the first die. Thus if $\mathcal{F}$ is your information set, you know whether or not

the first die showed something larger than the second die. None of the variables are measurable with respect to $\mathcal{F}$. For example on the event A X_1 could be anything between 2 and 6, X_1 could be anything between 1 and 5, while Y could be anything between 3 and 11.

(c) $\mathcal{F} = \sigma(X_1, Y)$.

Here you know the what the first die is showing and what the total across both dies is. Based in this information you can figure out what the second die is showing via $X_2 = Y - X_1$. If you know what each die is showing then you know exactly what happened, which means that $\mathcal{F}$ is actually the power set $\mathcal{P}(\Omega)$. Thus if $\mathcal{F}$ is your information set then you know everything, hence all variables are measurable with respect to $\mathcal{F}$ (indeed any random variable that could conceivably be defined on this sample space will be measurable with respect to the power set).

5. In lecture we considered the conditional expectation $E[X|\mathcal{F}]$, where X is the total number of tails obtained in ten tosses of a fair coin and $\mathcal{F}$ is the algebra generated by the outcomes of the first two tosses. We argued, on intuitive/heuristic grounds, that $E[X|\mathcal{F}] = Y + 4$. The point of this problem is to verify (partially, at least) that this random variable does satisfy the formal definition of conditional expectation.

   (a) Verify that $E[Y|A_i] = E[X|A_i]$ for $i = 1, 2, 3, 4$.

   Here is a concise way to solve the problem. Observe that $X = Y + (X - Y)$ and that $X - Y$ is the number of tails obtained over the final eight flips. Since the outcomes of the first two flips have no bearing on the outcomes of the final eight (and vice versa) we have that $P(X - Y = k|A_i) = P(X - Y = k)$ for any i and k; for *example* $P(X-Y=3|A_1) = P(X-Y=3) = \binom{8}{3}2^{-8}$*. Thus*

$$\begin{aligned} E[X-Y|A_i] &= \sum_{k=0}^{8} k\cdot P(X-Y=k|A_i) \\ &= \sum_{k=0}^{8} k\cdot P(X-Y=k) \\ &= E[X-Y] \\ &= 4\ , \end{aligned}$$

where we have used the fact that $X-Y$ *has a binomial distribution with parameters* $n=8$ *and* $p=0.5$*, which means that its expected value is* $np = 8(.5) = 4$*. Thus for any* i *we have*

$$\begin{aligned} E[X|A_i] &= E[Y+(X-Y)|A_i] \\ &= E[Y|A_i] + E[X-Y|A_i] \\ &= E[Y|A_i] + 4 \\ &= E[Y+4|A_i]\ . \end{aligned}$$

This approach essentially verifies the required property simultaneously for all values of i*. As an alternative we could go on a case-by-case basis as follows. For* A_2 *we have*

$$\begin{aligned} E[Y|A_2] &= 0\cdot P(Y=0|A_2) + 1\cdot P(Y=1|A_2) + 2\cdot P(Y=2|A_2) \\ &= 0\cdot 0 + 1\cdot 1 + 2\cdot 0 \\ &= 1\ , \end{aligned}$$

which means that $E[Y+4|A_2] = E[Y|A_2]+4 = 1+4 = 5$*. Now for* $1 \le k \le 9$ *we have (for other values of* k *the conditional probability is negative)*

$$\begin{aligned} P(X=k|A_2) &= P(k-1 \text{ of the remaining 8 flips are tails}) \\ &= \binom{8}{k-1}(.5)^{k-1}(.5)^{9-k} \\ &= \binom{8}{k-1}(.5)^{8}\ . \end{aligned}$$

Therefore

$$
\begin{aligned}
E[X|A_2] &= \sum_{k=0}^{10} k \cdot P(X = k|A_2) \\
&= 0 \cdot 0 + 1 \cdot \binom{8}{0}(.5)^8 + 2 \cdot \binom{8}{1}(.5)^8 + \ldots + 9 \cdot \binom{8}{8}(.5)^8 + 10 \cdot 0 \\
&= (0+1) \cdot \binom{8}{0}(.5)^8 + (1+1) \cdot \binom{8}{1}(.5)^8 + \ldots + (8+1) \cdot \binom{8}{8}(.5)^8 \\
&= \sum_{k=0}^{8}(k+1)\binom{8}{k}(.5)^8 \\
&= \sum_{k=0}^{8} k \cdot \binom{8}{k}(.5)^8 + \sum_{k=0}^{8} 1 \cdot \binom{8}{k}(.5)^8 \\
&= \sum_{k=0}^{8} k \cdot \binom{8}{k}(.5)^k(.5)^{8-k} + \sum_{k=0}^{8} \binom{8}{k}(.5)^k(.5)^{8-k} .
\end{aligned}
$$

The first sum is the expected value of a binomial random variable with parameters $n = 8$ *and* $p = 0.5$, *and is therefore equal to* 4. *The second sum is the sum over all possible values of binomial probability mass function, and therefore is equal to* 1. *Thus* $E[X|A_2] = 4 + 1 = 5 = E[Y + 4|A_2]$, *as required.*

(b) Verify that $E[Y|A_1 \cup A_2] = E[X|A_1 \cup A_2]$.

The inuition is as follows. I am only told that either A_1 *or* A_2 *occured, but I don't know exactly which one. This means that, on the basis of this information, I don't know the value of* Y *(but I have narrowed it down to either* 0 *or* 1*). If it was* A_1 *that occured then we would have* $Y = 0$ *and the probability it was* A_1, *given that it was either* A_1 *or* A_2, *is one half. Similarly if it was* A_2 *that occured then we would have* $Y = 1$ *and the probability it was* A_2, *given that it was either* A_1 *or* A_2, *is also one half. So* $E[Y|A_1 \cup A_2] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1/2$, *which means that* $E[Y + 4|A_1 \cup A_2] = 4.5$. *Moving on to* X. *If it was* A_1 *that occured then my best guess at the ultimate value of* X *is* $E[X|A_1] = 4$ *and that probability that*

it was A_1, given that it was either A_1 or A_2, is one half. And if it was A_2 that occured then my best guess at the ultimate value of X is $E[X|A_2] = 5$ and the probability that it was A_2, given that it was either A_1 or A_2, is also one half. Thus $E[X|A_1 \cup A_2] = 4 \cdot \frac{1}{2} + 5 \cdot 12 = 4.5$, as required.
The formal calculation underlying this is

$$E[X|A_1 \cup A_1] = E[X|A_1]P(A_1|A_1 \cup A_2) + E[X|A_2]P(A_2|A_1 \cup A_2) \ ,$$

etc. for Y. To see why this works observe that

$$\begin{aligned}
E[X|A_1 \cup A_2] &= \sum_{k=0}^{10} k \cdot P(X = k|A_1 \cup A_2) \\
&= \sum_{k=0}^{10} k \cdot \frac{P((X = k) \cap (A_1 \cup A_2))}{P(A_1 \cup A_2)} \\
&= \sum_{k=0}^{10} k \cdot \frac{P([(X = k) \cap A_1] \cup [(X = k) \cap A_2])}{P(A_1 \cup A_2)} \\
&= \sum_{k=0}^{10} k \cdot \frac{P((X = k) \cap A_1) + P((X = k) \cap A_2)}{P(A_1 \cup A_2)} \\
&= \sum_{k=0}^{10} k \cdot \frac{P((X = k) \cap A_1)}{P(A_1 \cup A_2)} + \sum_{k=0}^{10} k \cdot \frac{P((X = k) \cap A_2)}{P(A_1 \cup A_2)} \\
&= \sum_{k=0}^{10} k \cdot \frac{P(X = k|A_1)P(A_1)}{P(A_1 \cup A_2)} + \sum_{k=0}^{10} k \cdot \frac{P(X = k|A_2)P(A_2)}{P(A_1 \cup A_2)} \\
&= \frac{P(A_1)}{P(A_1 \cup A_2)} \sum_{k=0}^{10} k \cdot P(X = k|A_1) + \frac{P(A_2)}{P(A_1 \cup A_2)} \sum_{k=0}^{10} k \cdot P(X = k|A_2) \\
&= \frac{P(A_1)}{P(A_1 \cup A_2)} E[X|A_1] + \frac{P(A_2)}{P(A_1 \cup A_2)} E[X|A_2] \\
&= P(A_1|A_1 \cup A_2)E[X|A_1] + P(A_2|A_1 \cup A_2)E[X|A_2] \ ,
\end{aligned}$$

where we have used DeMorgan's Law and the fact that A_1 and A_2 are disjoint, as well as the definition of conditional probability. This level of detail is obviously not required for full marks.

(c) Verify that $E[X] = E[E[X|\mathcal{F}]]$.

There was a lot of confusion about this one. What I was looking for was for you to verify the tower property for this particular random variable X and field $\mathcal{F}$. We already know that $E[X|\mathcal{F}] = Y + 4$ and so all we need to confirm is that $E[X] = E[Y + 4]$. Now $E[X] = 5$ and $E[Y + 4] = E[Y] + 4 = 1 + 4 = 5$. Done.

If you are interested in the proof of the general result (i.e. for arbitrary random variables X and filtration $\mathcal{F}$) assume that $\mathcal{F}$ is generated by some partition $B_1, B_2, \ldots, B_n$. In general given a field one can always find a partition that generates it, and so this assumption can be made without loss of generality. $E[X|\mathcal{F}]$ is a random variable that takes on the value $E[X|B_i]$ on the event B_i. Using indicator random variables we can write

$$E[X|\mathcal{F}] = E[X|B_1]\mathbf{1}_{B_1} + E[X|B_2]\mathbf{1}_{B_2} + \ldots + E[X|B_n]\mathbf{1}_{B_n} \ .$$

Now $E[X|B_i]$ is a constant, whereas $\mathbf{1}_{B_i}$ is a random variable with $E[\mathbf{1}_{B_i}] = P(B_i)$. Thus

$$\begin{aligned}
E\left[E[X|\mathcal{F}]\right] &= E\left[E[X|B_1]\mathbf{1}_{B_1} + E[X|B_2]\mathbf{1}_{B_2} + \ldots + E[X|B_n]\mathbf{1}_{B_n}\right] \\
&= E\left[E[X|B_1]\mathbf{1}_{B_1}\right] + E\left[E[X|B_2]\mathbf{1}_{B_2}\right] + \ldots + E\left[E[X|B_n]\mathbf{1}_{B_n}\right] \\
&= E[X|B_1]E[\mathbf{1}_{B_1}] + E[X|B_2]E[\mathbf{1}_{B_2}] + \ldots + E[X|B_n]E[\mathbf{1}_{B_n}] \\
&= E[X|B_1]P(B_1) + E[X|B_2]P(B_2) + \ldots E[X|B_n]P(B_n) \ .
\end{aligned}$$

To see that this is the same as $E[X]$ first note that by the Law of Total Probability we can write

$$P(X = k) = \sum_{i=1}^{n} P(X = k|B_i)P(B_i) \ ,$$

and therefore

$$\begin{aligned}
E[X] &= \sum_k k \cdot P(X = k) \\
&= \sum_k k \cdot \sum_i P(X = k|B_i)P(B_i) \\
&= \sum_k \sum_i k \cdot P(X = k|B_i)P(B_i) \\
&= \sum_i \sum_k k \cdot P(X = k|B_i)P(B_i) \\
&= \sum_i P(B_i) \sum_k k \cdot P(X = k|B_i) \\
&= \sum_i P(B_i)E[X|B_i] \\
&= \sum_i E[X|B_i]P(B_i) \\
&= E[X|B_1]P(B_1) + E[X|B_2]P(B_2) + \ldots E[X|B_n]P(B_n) \ ,
\end{aligned}$$

as required.