MA370 - Summer 2013
Assignment #3

- Please work in groups of two.
- Due by Thursday, July 18, by 4:00 p.m. You may hand in your assignment at any time prior to the due date. My office is BA-546 - slip your assignment under the door if I am not there.

1. Let $\Omega = \{1, 2, 3, 4\}$. Which (if any) of the following collections are fields/algebras?
   (a) $\mathcal{F}_1 = \{\emptyset, \{1, 2\}, \{3, 4\}\}$.
   (b) $\mathcal{F}_2 = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}\}$.
   (c) $\mathcal{F}_3 = \{\emptyset, \Omega, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\}$.

2. Let $\Omega = [0, 1]$. Adding as few sets as possible, complete the family of sets $\{\emptyset, [0, 1/2), \{1\}\}$ to obtain a field on Ω.

3. Suppose I play three rounds of a game where, on each round, I either win or lose one dollar. Let Z_k denote my net winnings on round k for $k = 1, 2, 3$, so that Z_k is a random variable that either takes on the value 1 (if I win round k) or -1 (if I lose round k). Let $X_0 = 0$ and $X_n = \sum_{k=1}^{n} Z_k$ for $n = 1, 2, 3$, so that X_n denotes my total winnings after n rounds. Finally, let $M_n = \max(X_0, X_1, \ldots, X_n)$ for $n = 0, 1, 2, 3$.
   (a) Is X_3 is measurable with respect to $\sigma(M_3)$? If it is, explain how the value of X_3 can be deduced from that of M_3. If it is not, give an example of an event A such that $A \in \sigma(X_3)$ but $A \notin \sigma(M_3)$.

(b) Is Z_2 measurable with respect to $\sigma(X_2)$? If it is, explain how the value of Z_2 can be deduced from that of X_2. If it is not, give an example of an event A such that $A \in \sigma(Z_2)$ but $A \notin \sigma(X_2)$.

(c) Is Z_2 measurable with respect to $\sigma(X_1, X_2)$? If it is, explain how the value of Z_2 can be deduced from those of X_1 and X_2. If it is not, give an example of an event A such that $A \in \sigma(Z_2)$ but $A \notin \sigma(X_1, X_2)$.

4. Two fair dice are tossed, leading to a sample space

$$\Omega = \{(1,1), (1,2), \ldots, (1,6), (2,1), \ldots, (6,6)\} ,$$

where (i,j) indicates that i dots were showing on the first die and j dots were showing on the second die. Let X_k denote the number of dots showing on die k for $k = 1, 2$, and let $Y = X_1 + X_2$. Describe the information set corresponding to each of the following fields (i.e. answer the question "if $\mathcal{F}$ is your information set, then what do you know?"), and in each case determine which of the variables X_1, X_2, Y are measurable with respect to the given field.

(a) $\mathcal{F} = \sigma(B_1, \ldots, B_6)$, where $B_i = \{(i,1), (i,2), \ldots, (i,6)\}$.

(b) $\mathcal{F} = \sigma(A)$, where

$$\begin{aligned} A \ = \ & \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), \\ & (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\} \end{aligned}$$

(c) $\mathcal{F} = \sigma(X_1, Y)$.

5. In lecture we considered the conditional expectation $E[X|\mathcal{F}]$, where X is the total number of tails obtained in ten tosses of a fair coin and $\mathcal{F}$

is the algebra generated by the outcomes of the first two tosses. We argued, on intuitive/heuristic grounds, that $E[X|\mathcal{F}] = Y + 4$. The point of this problem is to verify (partially, at least) that this random variable does satisfy the formal definition of conditional expectation.

(a) Verify that $E[Y|A_i] = E[X|A_i]$ for $i = 1, 2, 3, 4$.

(b) Verify that $E[Y|A_1 \cup A_2] = E[X|A_1 \cup A_2]$.

(c) Verify that $E[X] = E[E[X|\mathcal{F}]]$.