MA370 - Summer 2013
Assignment #2 Solutions

1. In lecture on June 5 we considered an instance of the binomial model where the expected return from buying a put option was negative. In this problem we prove that this will be the case for *any* derivative that pays off more in the downstate than it does in the upstate (actually we prove that the excess return, or expected return on the derivative minus the risk-free rate, is negative). So for the purposes of this problem assume we are in the binomial world, let p denote the probability that the upstate is realized and consider a derivative with payoffs f_d and f_u.

   (a) Prove that $pf_u + (1-p)f_d < f_0(1+r)$ if and only if $f_u < f_d$, where f_0 is the fair price of the derivative.

   Using the risk-neutral pricing formula and letting q denote the risk-neutral probability that the stock goes up we have

   $$f_0 = \frac{f_u}{1+r}q + \frac{f_d}{1+r}(1-q) \ ,$$

   so that

   $$(1+r)f_0 = f_u q + (1-q)f_d \ .$$

   Thus the inequality given in the question is equivalent to

   $$pf_u + (1-p)f_d < f_u q + (1-q)f_d \ ,$$

   which, after a bit of rearranging, is equivalent to

   $$(p-q)(f_d - f_u) > 0 \ .$$

   Since $q < p$ (see lecture notes) implies that $p - q > 0$, it follows that the inequality holds if and only if $f_d > f_u$.

(b) Use (a) to show that if $f_d > f_u$ then the expected return on the derivative is less than r.

The expected return on the derivative is

$$\frac{f_u - f_0}{f_0} p + \frac{f_d - f_0}{f_0}(1-p) = \frac{pf_u + (1-p)f_d}{f_0} - 1 \, .$$

This return exceeds the risk-free rate if and only if

$$\frac{pf_u + (1-p)f_d}{f_0} - 1 > r \, ,$$

which is equivalent to

$$pf_u + (1-p)f_d > (1+r)f_0 \, .$$

Using (a) this will occur if and only if $f_d > f_u$. Therefore the expected excess return from selling a "down-state" derivative (i.e. one that pays off more in the down-state) is always positive, and the expected return from selling an "up-state" derivative (i.e. one that pays off more in the up-state) is negative.

(c) Provide a brief intuitive explanation for the result in (b).

Risk-averse investors are willing to pay a lot for securities that do well during bad times, to the extent that they are willing to accept a negative excess return. What I'm looking for here is some reference to (i) state value of money and/or (ii) the fact that "down-state" derivatives are effectively insurance policies against market downturns and/or (iii) a derivative that pays off more in the down than the up is insurance, and people are actually hoping to lose money on insurance.

2. The current price of a stock is \$100. The stock will either be worth \$110 or \$95 in one month. The yield on risk-free bonds maturing in one month is 1%.

(a) Determine the fair price of (i) a call option struck at \$105 and (ii) a put option struck at \$100.

The gross returns here are $u = 1.1$ *and* $d = 0.95$*. This gives us a risk-neutral probability of*

$$q = \frac{1.01 - 0.95}{1.1 - 0.95} = \frac{0.06}{0.15} = 0.4 \, .$$

The value of the call is therefore

$$\begin{aligned} c_0 &= \frac{\max(110-105,0)}{1.01}(.4) + \frac{\max(95-105,0)}{1.01}(.6) \\ &= \frac{5}{1.01}(.4) + \frac{0}{1.01}(.6) \\ &= \frac{2}{1.01} \\ &= 1.9801\ldots \, , \end{aligned}$$

and the value of the put

$$\begin{aligned} p_0 &= \frac{\max(100-110,0)}{1.01}(.4) + \frac{\max(100-95,0)}{1.01}(.6) \\ &= \frac{0}{1.01}(.4) + \frac{5}{1.01}(.6) \\ &= \frac{3}{1.01} \\ &= 2.9702\ldots \, . \end{aligned}$$

(b) How would you exploit a call struck at \$105 that was trading at \$4?

The call is (way) overpriced and so I'm going to sell it. In order to hedge my exposure and lock in the risk-free profit I'm going to need $\Delta = \frac{5-0}{110-95} = \frac{1}{3}$ *shares. The total cost of this position is* $\frac{100}{3} - 4$*, and if I borrow this amount at 1% my initial cash flow is zero while my cash flow at maturity is*

$$\frac{110}{3} - 5 - (1.01)\left[\frac{100}{3} - 4\right] = 2.04 = \frac{95}{3} - 0 - (1.01)\left[\frac{100}{3} - 4\right] \, .$$

Note that $2.04 = (1.01)\left[4 - \frac{2}{1.01}\right]$*, which is the future value of the mispricing.*

(c) How would you exploit a put struck at \$100 that was trading at \$1?

Since the put is underpriced I'm going to buy it. In order to hedge our exposure and lock in the risk-free profit recall that $\Delta = \frac{0-5}{110-95} = -\frac{1}{3}$ *would be the number of shares required to hedge a short position in the derivative; i.e. if we sold the put we'd need to short one-third of a share in order to hedge. Since we bought the derivative we're going to need to do the opposite, which means we're going to need a long position in* $\frac{1}{3}$ *shares for our hedge. The cost of the position is* $\frac{100}{3} - 1$*, and if we borrow this amount at 1% our initial cash flow is zero while our cash flow at maturity is*

$$\frac{110}{3} + 0 - (1.01)\left[\frac{100}{3} + 1\right] = 1.99 = \frac{95}{3} + 5 - (1.01)\left[\frac{100}{3} + 1\right].$$

Note that $1.99 = (1.01)\left[\frac{3}{1.01} - 1\right]$*, which is the future value of the mispricing.*

3. Consider a binomial model with $\mu = 0.12$, $r = 0.05$ and $p = 0.5$. Further suppose that the stock is currently trading at $S_0 = 50$.

(a) Plot the value of a call option on this stock as a function of σ for $K = 30$, $K = 50$ and $K = 70$. Comment on any differences and/or similarities between the plots. Please put all three plots on the same set of axes and use at least one hundred values of σ in the interval $(.05, .80)$.

Matlab code is posted on MLS and plots are given in Figure 1 towards the end of the solutions. Ignoring the flat spots for now, we observe that in all cases option value rises with volatility - call options love volatility (so do put options), regardless of their *"moneyness." In order to understand what is happening with the flat spots start with the out-of-the-money option with $K = 70$. If volatility is low there is no chance the option will rise far enough to end up in the money and it will always expire worthless - this leads to the flat region on the red curve. For the in-the-money option with $K = 30$ it is the exact opposite - when volatility is very low there is no chance the stock will fall far enough that it doesn't end up in the money. In order to prove that this leads to a flat spot in the graph observe that if $S_d > K$ in the binomial model (so that $S_u > K$ as well) then $c_u = S_u - K$ and $c_d = S_d - K$, leading to an option value of*

$$\begin{aligned} c_0 &= \frac{c_u}{1+r}q + \frac{c_d}{1+r}(1-q) \\ &= \frac{(S_u - K)q + (S_d - K)(1-q)}{1+r} \\ &= \frac{S_u q + S_d(1-q) - K}{1+r} \\ &= \left[\frac{S_u}{1+r}q + \frac{S_d}{1+r}(1-q)\right] - \frac{K}{1+r} \\ &= S_0 - \frac{K}{1+r}\ , \end{aligned}$$

where we have used the defining feature of the risk-neutral probabilities in the last line. So when volatility is low enough that the in-the-money option never expires without being exercised, its value is $50 - 30/1.05 = 21.4285\ldots$ as observed in the graph.

(b) Plot the delta of a call option on this stock as a function of σ for $K = 45$ (in the money option) and $K = 70$ (out of the money option). Comment on the shape of each graph individually, as well as any differences between the two. Please put each plot on a separate set of axes and, for each plot, use at least one hundred values of σ in the interval $(.05, .80)$.

Matlab code is posted on MLS (same script as part (a)), and plots are available in Figure 2 towards the end of the solutions. Asking for plots on separate axes was a typo and so I've put both plots on the same set of axes. In contrast to option value we see that option delta can either rise or fall with volatility and the critical factor is moneyness. For the out-of-the money option with low volatility there is no chance the option ends up being exercised, and therefore if I sell it (to some sucker) then I don't need to bother hedging - I have no exposure and so no need to insure against anything. As volatility rises, however, so does the probability that the option ends up being exercised and if I'm short the option I need to start hedging my exposure to upward movements in the stock. It's the exact opposite for the in-the-money option. If the option begins in the money and volatility is extremely low, the option is guaranteed to be exercised and I'm going to need maximum insurance. To see why $\Delta = 1$ *in this case observe that* $\Delta = \frac{(S_u-K)-(S_d-K)}{S_u-S_d} = 1$ *whenever* $S_d > K$. *As volatility rises it becomes possible that the option ends up not being exercised, and I can ease back on how much insurance coverage I need.*

4. Consider the trinomial example discussed in lecture over June 10 and June 12. Instead of the call struck at \$60, suppose that I am interested in introducing a put struck at \$48.

   Note: When you encounter numbers of the form $\frac{x}{1.02}$ in this problem I would *much* prefer that you leave it in that form instead of carrying out the division.

   (a) Write the price of the put in terms of the middle-state price Ψ_m.

   Let p_i *denote the value of the put in state* i *for* $i = u, m, d$. *Then* *price of the put is*

$$
\begin{aligned}
\Psi_u \cdot p_u + \Psi_m \cdot p_m + \Psi_d \cdot p_d &= \Psi_u \cdot 0 + \Psi_m \cdot 0 + \Psi_d \cdot 6 \\
&= \left[\frac{0.625}{1.02} - \frac{2}{3}\Psi_m\right] \cdot 6 \\
&= \frac{3.75}{1.02} - 4\Psi_m
\end{aligned}
$$

where we have used the relation $\Psi_d = \frac{0.625}{1.02} - \frac{2}{3}\Psi_m$ *that was derived in lecture.*

(b) Determine the (i) no-arbitrage and (ii) economically reasonable no-arbitrage intervals for the price of the put.

In lecture the no-arbitrage interval for the middle-state price was found to be $(0, \frac{0.9375}{1.02})$ *and the economically reasonable no-arbitrage interval was found to be* $(\frac{0.28125}{1.02}, \frac{0.375}{1.02})$. *Plugging in the endpoints of each interval into the put-price formula found in (a), we get an arbitrage-free interval of*

$$
\left(\frac{3.75}{1.02} - 4 \cdot \frac{0.9375}{1.02}, \frac{3.75}{1.02} - 4 \cdot 0\right) = \left(0, \frac{3.75}{1.02}\right) \ ,
$$

so any price that is less than (approximately) \$3.68 will keep the market free of arbitrage. It would seem absurd for this option to trade at one penny, and so now let's turn to the economically reasonable interval to see what we would actually expect to see the thing trade at:

$$
\left(\frac{3.75}{1.02} - 4 \cdot \frac{0.375}{1.02}, \frac{3.75}{1.02} - 4 \cdot \frac{0.28125}{1.02}\right) = \left(\frac{2.25}{1.02}, \frac{2.6205}{1.02}\right) \ .
$$

So I would expect to see the put trade anywhere between (approximately) \$2.21 and \$2.57.

(c) Suppose now that I am interested in introducing a call option struck at \$60 *and* a put option struck at \$48 (if it helps assume that I'm going to introduce these derivatives simultaneously and sell them to different clients). If my boss instructs me to price the call at c_0, then how should I price the put? In particular, am I still free to pick any price in the interval from (b)?

In lecture we saw that the price of the call had to satisfy

$$c_0 = \frac{2.25}{1.02} - 2\Psi_m ,$$

and in part (a) we saw that the price of the put had to satisfy

$$p_0 = \frac{3.75}{1.02} - 4\Psi_m .$$

Isolating Ψ_m in the first equation and plugging the result into the second equation we see that the prices of the put and call must be related as follows

$$p_0 = 2c_0 - \frac{0.75}{1.02} .$$

So, given my instructions, I am not free to pick any price from the interval in (b); for example if my boss tells me to sell the call for \$1.50 then I'm going to have to sell the put for about \$2.26.

5. Consider a trinomial model with $u = 1.55$, $m = 1.10$ and $d = 0.65$. Further assume that $r = 0.04$, $p_u = p_d = 0.2$ and $p_m = 0.6$.

   (a) Calculate the expected value and standard deviation of the stock's return.

   The returns in each state are $u-1$, $m-1$ and $d-1$. The expected return is therfore

   $$(.2)(.55) + (.6)(.1) + (.2)(-.35) = 0.1 ,$$

   and the standard deviation of the return is

   $$\sqrt{(.2)(.55-.1)^2 + (.6)(.1-.1)^2 + (.2)(-.35-.1)^2} ,$$

which simplifies to

$$\sqrt{2(.2)(.45)^2} = (.45)\sqrt{.4} = .2846\ldots\ .$$

So the average return is 10% *on a volatility of approximately* 28.5%*.*

(b) Find the general form of a state price vector in this market, parametrized by the price of a middle-state dollar.

Using the formulae from lecture (or page 29 of the notes) we get that state prices must satisfy

$$\Psi_u = \frac{(1+r)-d}{(1+r)(u-d)} - \frac{m-d}{u-d}\Psi_m = \frac{.39}{(1.04)(.9)} - (0.5)\Psi_m$$

and

$$\Psi_d = \frac{u-(1+r)}{(1+r)(u-d)} - \frac{u-m}{u-d}\Psi_m = \frac{.51}{(1.04)(.9)} - (0.5)\Psi_m\ .$$

(c) What range of middle-state prices is consistent with no arbitrage?

In order to ensure that $\Psi_u > 0$ *we must have*

$$\Psi_m < \frac{.39}{(1.04)(.9)(.5)} = 0.8333\ldots\ ,$$

and in order that $\Psi_d > 0$ *we must have*

$$\Psi_m < \frac{.51}{(1.04)(.9)(.5)} = 1.0897\ldots\ .$$

Clearly both of these are satisfied provided $\Psi_m < 0.8333\ldots$*, and since we must also have* $\Psi_m > 0$ *the range of middle-state prices that is consistent with no-arbitrage is* $(0, 0.8333\ldots)$*.*

(d) What range of middle-state prices is consistent with no arbitrage *and* risk-aversion?

Now we have to enforce $\Psi_d > \Psi_m > \Psi_u$*. Since*

$$\Psi_d = \frac{.51}{(1.04)(.9)} - (0.5)\Psi_m$$

this will happen if and only if

$$\frac{.51}{(1.04)(.9)} - (0.5)\Psi_m > \Psi_m \ ,$$

or

$$\Psi_m < \frac{.51}{(1.04)(.9)(1.5)} = 0.3632\ldots \ .$$

And in order to enforce $\Psi_u < \Psi_m$ *we must ensure that*

$$\frac{.39}{(1.04)(.9)} - (0.5)\Psi_m < \Psi_m \ ,$$

or

$$\Psi_m > \frac{.39}{(1.04)(.9)(1.5)} = 0.2777\ldots \ .$$

So the arbitrage-free-and-risk-averse interval is $(0.2777\ldots, 0.3632\ldots)$, *which is much tighter than the arbitrage-free interval.*

(e) Determine the arbitrage-free price range for a put struck at \$25, assuming the stock is currently trading at \$20, as well as the "economically reasonable" arbitrage-free price range.

The stock will either be worth \$13, \$22 or \$31. Therefore the put will either be worth $\bar{p}_d = 12$ *in the down-state,* $\bar{p}_m = 3$ *in the mid-state and* $\bar{p}_u = 0$ *in the up-state. The general form for the value of the put in terms of the parameter* Ψ_m *is*

$$\begin{aligned}
p_0 &= p_0(\Psi_m) \\
&= \bar{p}_u\Psi_u + \bar{p}_m\Psi_m + \bar{p}_d\Psi_d \\
&= 0 \cdot \left[\frac{.39}{(1.04)(.9)} - (0.5)\Psi_m\right] + 3 \cdot \Psi_m + 12 \cdot \left[\frac{.51}{(1.04)(.9)} - (0.5)\Psi_m\right] \\
&= \frac{12(.51)}{(1.04)(.9)} - 3\Psi_m \ .
\end{aligned}$$

The maximum and minimum values for the put are obtained by plugging in the minimum and maximum values of Ψ_m *respectively. For the arbitrage-free range we plug in the endpoints of* $(0, 0.8333\ldots)$ *to get a price range of approximately* $(4.04, 6.54)$. *For the risk-averse arbitrage-free range we plug in the endpoints of* $(0.2777\ldots, 0.3632\ldots)$ *to get an approximate price range of* $(5.45, 5.71)$.

(f) Suppose that you sell the call to a client for \$5.60 and hedge your exposure by purchasing δ shares (recall that if $\delta < 0$ the "purchase" is actually a short sale). On the same set of axes plot your profit-and-loss in each state as a function of δ, and explain why it is not possible to eliminate all of the risk associated with selling the option. For the plots please use at least 100 points in the range $(-1, 0)$.

Recall that the profit/loss is

$$PL(\delta) = \delta(S_T - (1+r)S_0) - [p_T - (1+r)p_0] \ ,$$

where we use S_T *to denote the (random) terminal stock price and* p_T *the (random) terminal value of the put. In the various states this*

$$\begin{aligned}
PL_u(\delta) &= \delta(31 - (1.04)20) - [0 - (1.04)5.60] \\
&= 10.20\delta + 5.824 \\
PL_m(\delta) &= \delta(22 - (1.04)20) - [3 - (1.04)5.60] \\
&= 1.2\delta - 2.824 \\
PL_d(\delta) &= \delta(13 - (1.04)20) - [12 - (1.04)5.60] \\
&= -7.8\delta - 6.176
\end{aligned}$$

The plots can be found in Figure 3 and there is clearly no value of δ *that will ensure profit/loss is positive in every state. For* *example in order to ensure that we do not lose money in the up-state we must have* $\delta \leq -5.824/10.2 = -0.57\ldots$. *If* δ *is this small, however, profit/loss in the down-state is no larger than* $-7.8(-0.57\ldots) - 6.176 = -1.72\ldots$. *Hence if we protect against loss in the up-state we ensure a loss in the down-state.*

(g) Determine the hedge ratio (i.e. value of δ) that minimizes the volatility (i.e. standard deviation) of your profit-and-loss. How much does hedging reduce volatility relative to selling a naked (i.e. unhedged) put.

The mean and standard deviation of the terminal stock price are \$22 *and* \$5.692..., *respectively. The corresponding quantities for the terminal payoff of the put are* \$4.2 *and* \$4.06..., *respectively. Notice how much larger the standard deviation of the derivative is, relative to its expected value, compared with the underlying. The correlation between the two is* $-0.9325\ldots$. *See the posted Matlab script for calculation details. Using the formula given in lecture the optimal hedge ratio is therefore*

$$\delta^* = -\frac{2}{3},$$

so we'll need to short two-thirds of one share in order to hedge our position. Plugging this value into the expression for the variance of profit/loss given in lecture we get a volatility of $\mathrm{SD}(\delta^*) = 1.4696\ldots$, *which is approximately 36% of the volatility of the derivative alone (i.e. the volatility of selling a naked put). Thus hedging reduces volatility by nearly two-thirds.*

Figure 1: Plots for 3(a)

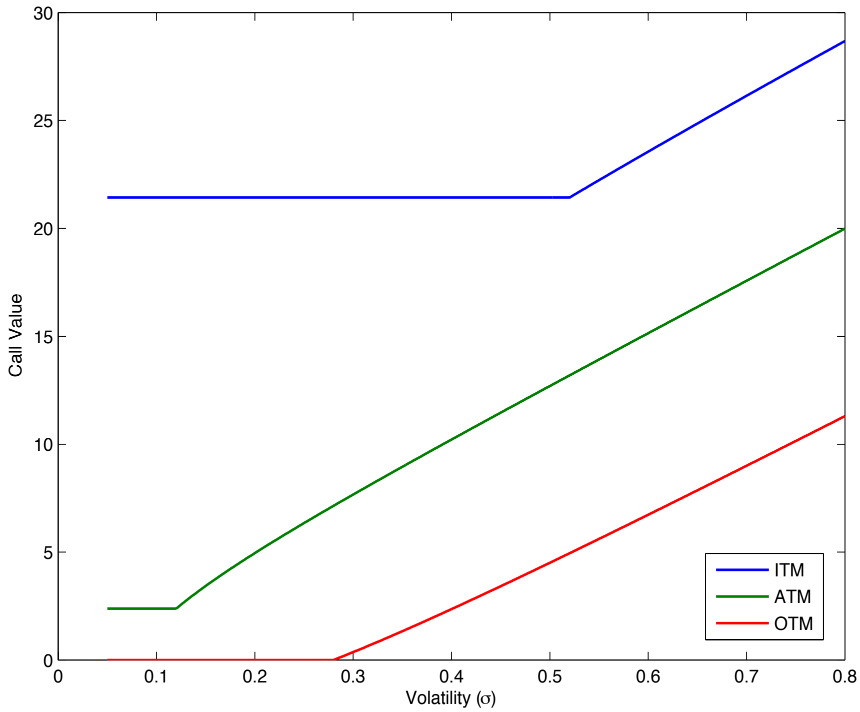

Figure 2: Plots for 3(b)

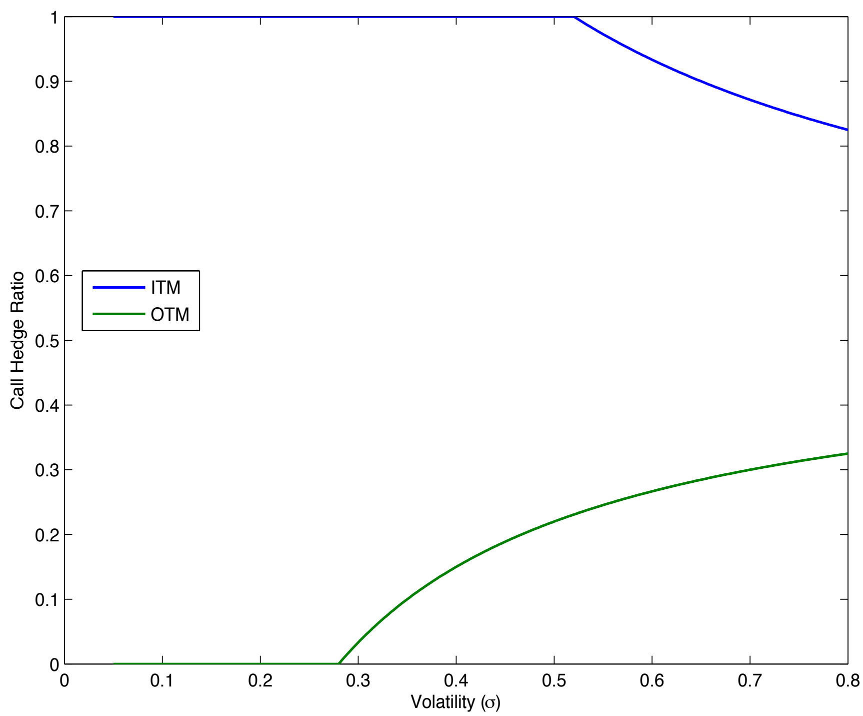

Figure 3: Profit/Loss Graph for Question 5(f).