MA370 - Summer 2013

Assignment #1 Solutions

1. Consider the markets defined by the payoff matrices

$$D_1 = \begin{bmatrix} 100 & 100 & 100 \\ 70 & 110 & 150 \end{bmatrix} , \quad D_2 = \begin{bmatrix} 100 & 100 & 100 \\ 70 & 110 & 150 \\ 10 & 50 & 90 \end{bmatrix} .$$

   (a) Explain why the first market is incomplete. Are there any redundant assets in this market?

   The market is incomplete because the number of assets is less than the number of states of nature. There are no redundant assets, because the rank of the matrix is equal to the number of assets.

   (b) Explain why the second market is incomplete. Determine the number of redundant assets, and for each redundant asset that you identify describe how its payoff can be replicated by trading in the other securities.

   The rank of the payoff matrix is two and there are three assets. Therefore the market is incomplete and there is one (number of assets less rank of payoff matrix) redundant asset. In order to identify a redundant asset we need to solve

$$\phi_1 \mathbf{S}_1 + \phi_2 \mathbf{S}_2 + \phi_3 \mathbf{S}_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} ,$$

   where $\mathbf{S}_i$ *is the* i^{th} *row of the payoff matrix. This is equivalent to solving*

$$D^T \phi^T = \mathbf{0} ,$$

   where $\mathbf{0}$ *is a three-by-one column of zeros. The reduced row echelon form of the relevant adjoint matrix is*

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.6 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] .$$

Thus any portfolio with $\phi_1 = 0.6\phi_3$ *and* $\phi_2 = -\phi_3$ *will have zero payoff in all states of nature. In particular if we set* $\phi_3 = -1$ *we see that the portfolio* $\phi = \begin{bmatrix} -0.6 & 1 & 0 \end{bmatrix}$ *will have zero payoff, equivalently*

$$-\frac{3}{5}\mathbf{S}_1 + \mathbf{S}_2 = \mathbf{S}_3 \ ,$$

and we see that the third asset is effectively a short position in $3/5$ *units of the first asset combined with a long position in one unit of the second asset.*

(c) For each market, find all portfolios that have a terminal value of $\mathbf{v} = [20 \quad 10 \quad 0]$.

In either case we're looking for all solutions to $\phi D_i = \mathbf{v}$, *or* $D_i^T \phi^T = \mathbf{v}^T$. *The reduced row echelon form of the adjoint matrix* $\left[D_1^T \;\middle|\; \mathbf{v}^T \right]$ *is*

$$\left[\begin{array}{cc|c} 1 & 0 & 0.375 \\ 0 & 1 & -0.25 \end{array}\right] ,$$

which means that the portfolio $\phi = \begin{bmatrix} 0.375 & -0.25 \end{bmatrix}$ *is the only portfolio that has a terminal value of* $\mathbf{v}$. *The reduced row echelon form of the adjoint matrix* $\left[D_2^T \;\middle|\; \mathbf{v}^T \right]$ *is*

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.6 & 0.375 \\ 0 & 1 & 1 & -0.25 \\ 0 & 0 & 0 & 0 \end{array}\right] ,$$

which means that any portfolio of the form

$$\begin{bmatrix} 0.375 & -0.25 & 0 \end{bmatrix} + \begin{bmatrix} 0.6 & -1 & 1 \end{bmatrix} v_3 \ , \qquad\qquad v_3 \in \mathbb{R}$$

has terminal payoff $\mathbf{v}$.

(d) What does your answer from (c) tell you about the presence of redundant assets and uniqueness of replicating portfolios?

In general, attainable payoffs will have unique replicating portfolios if there are no redundant assets, but not if there are redundant assets. In other words the presence of redundant assets means that any attainable payoff can be replicated in more than one way.

2. Refer to the market defined by the payoff matrix D_1 in Question 1.

   (a) Find a basis for the set of attainable payoffs.
   The reduced row echelon form of D_1 is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix},$$

   which means that a basis for the column space of D_1, i.e. a basis for the set of attainable payoffs, is the set

$$\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\}.$$

   Recall that this means any attainable payoff is a linear combination of these two vectors, and is therefore of the form

$$\phi_1 \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \phi_2 \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & 2\phi_2 - \phi_1 \end{bmatrix}.$$

   In particular if $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ *attainable, then it must the the case that* $v_3 = 2v_2 - v_1$.

   (b) Give an example of a payoff that is not attainable in this market.
   Any vector $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ *such that* $v_3 \neq 2v_2 - v_1$ *will be unattainable. So* $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ *is unattainable.*

   (c) Suppose your client is interested in a portfolio that is worth v_1 in state 1 and v_2 in state 2. Does your client have any freedom to choose the value of the portfolio in state 3? If so, explain how. If not, determine the implied state 3 payoff as a function of v_1 and v_2.
   No the client does not have any freedom, the state three payoff must be $2v_2 - v_1$.

(d) Now suppose your client is interested in a portfolio that is worth v_1 in state 1 and v_3 in state 3. Does your client have any freedom to choose the value of the portfolio in state 2? If so, explain how. If not, determine the implied portfolio value in state 2 in terms of v_1 and v_3.

Since $v_3 = 2v_2 - v_1$ *for any attainable payoff it follows that* $v_2 = (v_1 + v_3)/2$. *So no the client does not have any freedom, the state two payoff must be the average of the states one and three payoffs.*

(e) What do your answers from (c) and (d) tell you about the relation between the rank of the payoff matrix and the number of states that you can "choose your payoff" in.

They are the same. You can pick your payoff in $\text{rank}(D)$ *states, and the payoff in the remaining* $M - \text{rank}(D)$ *states is then implied.*

(f) Find a basis for the set of attainable payoffs in the market defined by the payoff matrix D_2. Compare this basis with the one obtained in (a), commenting on any similarities or differences.

The reduced row echelon form of D_2 *is*

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

and so the basis for the set of attainable payoffs is identical to that in the first market. In other words the set of attainable payoffs is identical.

3. Consider a market defined by the payoff matrix and initial price vectors

$$D = \begin{bmatrix} 100 & 100 & 100 \\ 70 & 110 & 150 \\ 150 & 100 & 80 \end{bmatrix}, \qquad S_0 = \begin{bmatrix} 95 \\ 84.5 \\ 123 \end{bmatrix}$$

(a) Show that the market is complete and free of arbitrage.

The determinant of D *is* $120,000$,*which means that the matrix is* *invertible (hence it has full rank) so the market is complete. The unique state price vector is*

$$\Psi = D^{-1}S_0 = \begin{bmatrix} 0.6 & 0.25 & 0.1 \end{bmatrix}^T .$$

This vector is strictly positive and so the market is free of arbitrage.

(b) Find the portfolio that has terminal value $\mathbf{v} = \begin{bmatrix} 150 & 100 & 150 \end{bmatrix}$.
The portfolio is $\phi^{(\mathbf{v})} = \mathbf{v}D^{-1} = \begin{bmatrix} -5.5416\ldots & 2.916\ldots & 3.333\ldots \end{bmatrix}$

(c) What is the cost of the portfolio from (b)?
The cost is $\phi^{(\mathbf{v})}S_0 = 130$.

(d) Calculate $\mathbf{v}\Psi$, where Ψ is the unique state price vector for this market.
$\mathbf{v}\Psi = 150(.6) + 100(.25) + 150(.1) = 130.$

(e) Comment on any similarities between your answers from (b)/(c) and (d).
The answers are exactly the same. The point of this problem is to illustrate that there are effectively to ways to find the price of an attainable payoff. The first is to (i) find the replicating portfolio and then (ii) determine its cost. The second is to simply use the state prices - this method is much faster since we don't need to bother with the replicating portfolio.

4. In a complete market with no redundant assets the payoff matrix is guaranteed to be invertible. Prove that in such a market the cost of a portfolio with a terminal value of $\mathbf{v}$ is equal to $\mathbf{v}\Psi$, where Ψ is the market's unique state price vector.
If there are no redundant assets and the market is complete then there is exactly one portfolio with terminal value $\mathbf{v}$*, namely* $\phi^{(\mathbf{v})} = \mathbf{v}D^{-1}$*. The cost of this portfolio is*

$$\phi^{(\mathbf{v})}S_0 = (\mathbf{v}D^{-1})S_0 = \mathbf{v}(D^{-1}S_0) = \mathbf{v}\Psi ,$$

as required.

5. Consider the market defined by the payoff matrix

$$D = \begin{bmatrix} 100 & 100 & 100 \\ 70 & 110 & 150 \end{bmatrix}$$

(a) Show that if $S_0 = \begin{bmatrix} 50 \\ 90 \end{bmatrix}$ the market admits arbitrage.

In order to find all state price vectors we must solve $S_0 = D\Psi$. *The reduced row echelon form of the adjoint matrix* $\begin{bmatrix} D & \big| & S_0 \end{bmatrix}$ *is*

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -0.875 \\ 0 & 1 & 2 & 1.375 \end{array}\right]$$

This means that any state price vector must have $\Psi_1 = \Psi_3 - 0.875$ *and* $\Psi_2 = 1.375 - 2\Psi_3$. *In order to ensure that* $\Psi_1 > 0$ *we must have* $\Psi_3 > 0.875$. *But if* $\Psi_3 > 0.875$ *then* $\Psi_2 = 1.375 - 2\Psi_3 < -0.375$. *Thus there are no state price vectors that are strictly positive, and the market therefore admits arbitrage.*

(b) Find the portfolio that has a terminal value of $\mathbf{v} = \begin{bmatrix} 1.375 & 0.875 & 0.375 \end{bmatrix}$ and show that it is an arbitrage portfolio. What do you think would happen to the prices of the two assets in a well-functioning market?

The replicating portfolio is $\phi = \begin{bmatrix} 0.0225 & -0.0125 \end{bmatrix}$, *its payoff is* $\mathbf{v}$ *and its cost is zero. So it is free money. In a well-functioning market I would expect the price of the first asset to get bid up (the arbitrage involves buying the first asset) and the price of the second asset to get bid down (the arbitrage involves selling the first asset).*

(c) Show that if $S_0 = \begin{bmatrix} 90 \\ 75 \end{bmatrix}$ then the market is free of arbitrage.

With these prices we get a reduced adjoint matrix of

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0.6 \\ 0 & 1 & 2 & 0.3 \end{array}\right],$$

which means that $\Psi_1 = \Psi_3 + 0.6$ *and* $\Psi_2 = 0.3 - 2\Psi_3$. *So the market is free of arbitrage provided* Ψ_3 *lives somewhere between* 0 *and* 0.15.

(d) If prices are as in part (c), determine upper and lower bounds on state prices for each state.

Ψ_1 *will lie somewhere between* 0.6 *and* 0.75, *while* Ψ_2 *will lie somewhere between* 0 *and* 0.3. *So the first state is highly valued relative to the other states.*

6. Consider the market defined by the payoff matrix and initial price vector

$$D = \begin{bmatrix} 100 & 100 & 100 \\ 70 & 110 & 150 \\ 150 & 100 & 80 \end{bmatrix}, \qquad S_0 = \begin{bmatrix} 95.0 \\ 84.5 \\ 123 \end{bmatrix}.$$

(a) Show that the market is complete and free of arbitrage.

Done in 3(a).

(b) Use the state prices found in (a) to determine (i) the implied risk-free rate of interest and (ii) the risk-neutral probabilities of each state.

Implied risk-free rate solves

$$\Psi_1 + \Psi_2 + \Psi_3 = \frac{1}{1+r},$$

so $r = (1/.95) - 1 = .0526\ldots$, *or approximately 5.26%. The risk neutral probabilities are* $q_j = (1+r)\Psi_j$, *so*

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 0.6315\ldots & 0.2631\ldots & 0.1052\ldots \end{bmatrix}$$

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(c) If the risk-neutral probabilities are used, calculate the expected return on the second and third assets.

Using the risk-neutral probabilities we get that the expected return on the second asset is

$$\begin{aligned} & \frac{70-84.5}{84.5}q_1+\frac{110-84.5}{84.5}q_2+\frac{150-84.5}{84.5}q_3 \\ \approx\ & (-.1716)(.6315)+(.3017)(.2631)+(.7751)(.1052) \\ =\ & 0.05255\ldots\ , \end{aligned}$$

and had I not rounded off the returns and risk-neutral probabilities, this would be exactly equal to the risk-free rate of interest. For the third asset the calculation is

$$\begin{aligned} & \frac{150-123}{123}q_1+\frac{100-123}{123}q_2+\frac{80-123}{123}q_3 \\ \approx\ & (.2195)(.6315)+(-.1870)(.2631)+(-.3496)(.1052) \\ =\ & 0.05264\ldots\ , \end{aligned}$$

and again, if I hadn't rounded this would be exactly equal to the risk-free rate.

(d) If the actual probabilities of each state are $p_1 = .1$, $p_2 = .8$ and $p_3 = .1$, determine the expected return on all three assets. The return you get for the third asset should snap your head back. What is going on there?

The basic idea here is to replace the risk-neutral probabilities q_j with the actual probabilities p_j. If we define $p = \begin{bmatrix} .1 & .8 & .1 \end{bmatrix}^T$ we can calculate all three returns in one-line using Matlab by inputting $(D{*}p{-}S_0)./S_0$. The results are $100r\%$, approximately 30.17% and approximately -16.26%. The reason for the very negative return on the third asset is that it pays handsomely in a very desirable state of nature (desirable because the state price for that state is so high). It is so desirable to have money in that state people are willing to lose money on average for the chance at a big payoff in the desirable state.