MAT 1330, Fall 2013 Solution to Assignment 1 (with marking scheme)

Total = 10 marks

***Question* 1**. In one of the forests of Alberta, every year 20% of the population of red deer are either dead or eaten by wild animals, and 1000 new red deer will join the group. The discrete-time dynamical system that gives the population of red deer each year is $p(t + 1) = 0.8p(t) + 1000$.

(a) If there are 2000 red deer now, how many red deer will be there three years later?

Answer. After three years, there will be **3464** red deer. [0.5]

Work. $p(1) = 0.8 \times 2000 + 1000 = 2600$, $p(2) = 0.8 \times 2600 + 1000 = 3080$, $p(3) = 0.8 \times 3080 + 1000 = 3464$.

(b) Give the updating function of the dynamical system. Find its inverse, if it exists.

Answer. The updating function of this system is $f(x) =$ **$0.8x + 1000$**. [0.5]

The inverse of the updating function is $f^{-1}(x) =$ **$1.25(x - 1000)$**. [0.5]

Work. $y = 0.8x + 1000$, $0.8x = y - 1000$, $x = (y - 1000) / 0.8$.
The inverse is $y = 1.25(x - 1000)$.

(c) Determine equilibrium point(s) of the dynamical system.

Answer. The equilibrium point(s) is $p^* =$ **5000** . [0.5]

Work. Let $x = 0.8x + 1000$. $0.2x = 1000$, $x = 5000$.

(d) Find the general solution of the dynamical system (i.e., a formula in terms of t) given the initial condition $p(0) = 2000$.

Answer. The general solution of this system is **$p(t) = -3000 \times 0.8^t + 5000$**. [2]

Work. The general formula can be obtained in a number of ways:

(i) Use the formula $p(t) = (x_0 - x^*)r^i + x^*$. $x_0 = 2000$, $x^* = 5000$, $r = 0.8$, $p(t) = -3000 \times 0.8^t + 5000$.

(ii) Find the pattern of the solution:

$p(1) = 2000 \times 0.8 + 1000$,
$p(2) = 2000 \times 0.8^2 + 1000 \times 0.8 + 1000$,

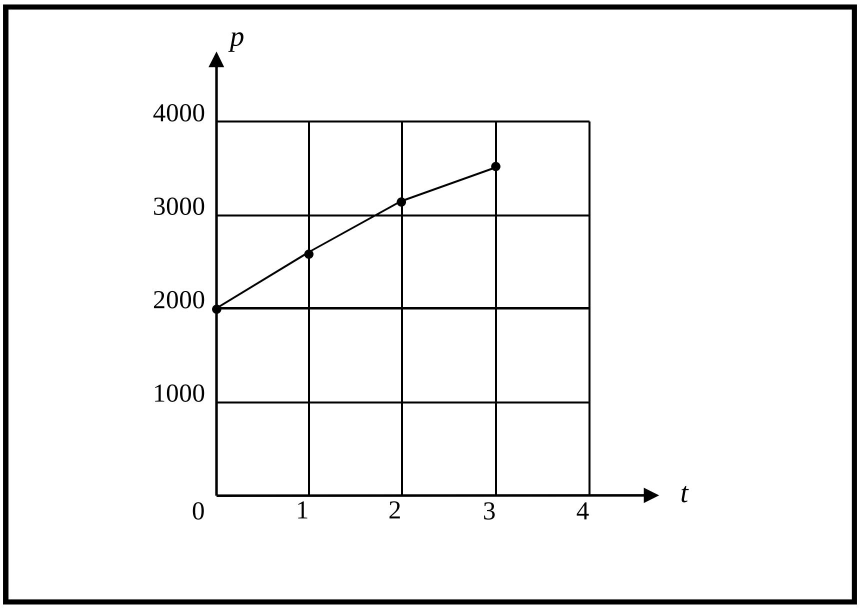
$p(3) = 2000 \times 0.8^3 + 1000 \times 0.8^2 + 1000 \times 0.8 + 1000,$
$\ldots .$

$p(t) = 2000 \times 0.8^t + 1000 \times (0.8^{t-1} + 0.8^{t-2} + \ldots + 1) = 2000 \times 0.8^t + 1000 \times \dfrac{1-0.8^t}{1-0.8}$
$=2000 \times 0.8^t + 5000 \times (1 - 0.8^t) = -3000 \times 0.8^t + 5000.$

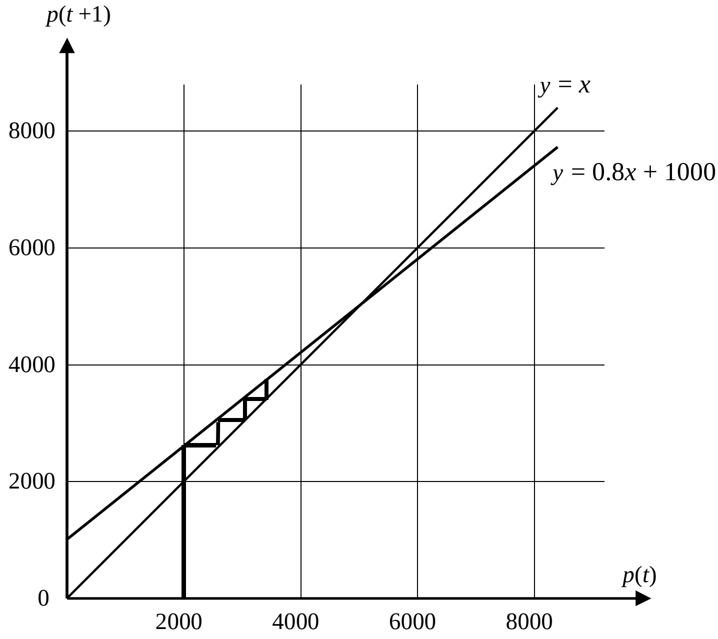
(iii) Use the observation that the general solution is of the form $p(t) = ar^t + b$. Here, $r = 0.8$.

$p(0) = a + b = 2000$, $p(1) = 0.8a + b = 2600$. Thus, $0.2a = -600$, $a = -3000$. $b = 2000 - a = 5000$. Hence, $p(t) = -3000 \times 0.8^t + 5000$.

(e) Draw the graph of the solution of the dynamical system with $p(0) = 2000$ with point $p(t)$, $t = 0, 1, 2, 3$. [1]



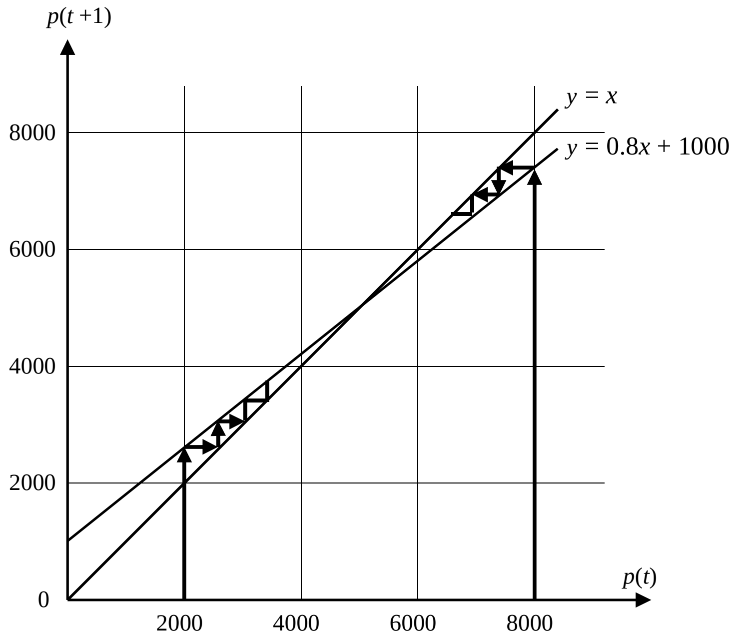
(f) Draw the cobweb diagram of the first four iterations of the dynamical system with $p(0) = 2000$. [1]



(g) Determine the stability of the equilibrium point(s). Justify your answer by the cobweb diagram showing how $p(t)$ changes when t increases with different initial values.

Answer. **<u>The only equilibrium point $p^* = 5000$ is stable.</u>** [0.5]

The cobwebbing graph: [0.5]



***Question* 2.** Assume that the dynamics of caffeine absorptions is modeled by a discrete-time dynamical system with dynamic rule $C(t + 1) = 0.87C(t)$, where t is time in hours. How many hours are needed to have at least 80% of the caffeine eliminated from the body?

Answer. At least 80% caffeine will be eliminated from the body after **12** hours.

Work.

The general solution of this system is $C(t) = 0.87^t C_0$. When $C(t) \le (1 - 0.8)C_0$,

$0.87^t C_0 \le 0.2C_0$, $0.87^t \le 0.2$. Hence, $t \ln 0.87 \le \ln 0.2$. Since $\ln 0.87 < 0$, $t \ge \ln 0.2 / \ln 0.87 \approx 11.56$. $t = 12$.

***Question* 3**. Consider the discrete-time dynamical system with the following dynamic rule:

$$N(t+1) = \frac{rN(t)}{N(t)-3},$$

where r is a parameter, $r \ne 0$, and $N(t)$ may take any real value except $N(t) \ne 3$. Find the value(s) (if they exist) of the parameter r for which

(a) the system has no equilibria,
(b) the system has exactly one equilibrium, and
(c) the system has nonnegative equilibria.

Answer. (a) **None**. [1]

(b) $\mathbf{r = -3}$. [1]

(c) $\mathbf{r \ge 3}$.

Work.

The equilibrium point satisfies the equation $x = \dfrac{rx}{x-3}$. Hence, $x(x - 3) - rx = 0$, $x(x - 3 - r) = 0$. Then $x = 0$, $x = r + 3$. [1]

(a) For any value r, this system has at least one equilibrium point $N^* = 0$. There is no value r for which the system does not have equilibria.

(b) When $r = -3$, $r + 3 = 0$ and this system has exactly one equilibrium $N^* = 0$.

(c) When $r \ge 3$, $r + 3$ is nonnegative.