Midterm, MATH 251

There are five regular problems, each one worth six points, for a total of 30 points. There is also one bonus problem, worth three points. No notes are allowed. An approved calculator is allowed (but unnecessary).

1.

W is the subspace of $\mathbb{R}^4$ given by $W = \{(a, b, c, d) : a + b + c = 0, d = -b\}$. Find a basis for W.

2.

T is the linear transformation from $P_2(\mathbb{R})$ to $P_2(\mathbb{R})$ given by

$$T(p(x)) = xp'(x)$$

where $p(x) \in P_2(\mathbb{R})$ and $p'(x)$ is the derivative of $p(x)$.
(a) Determine the nullity and the rank of T. (4 points)
(b) Is T onto? (1 point)
(c) Is T one-to-one? (1 point)

3.

The *trace* $\operatorname{tr}(A)$ of an $n \times n$ matrix A is the sum of the diagonal entries of A. Show that the trace is a linear transformation from $M_{n\times n}(\mathbb{R})$ to $\mathbb{R}$.

4.

T is the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 2a - b \\ a + b \end{pmatrix}$$

Find the matrix representation of T with respect to the ordered bases α and β, where α is the standard ordered basis for $\mathbb{R}^2$ and

$$\beta = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

5.

T is a linear transformation from a vector space V to V. Show that $R(T^2)$ is a subspace of $R(T)$.

Bonus problem (3 points):

$T : \mathbb{R}^2 \to \mathbb{R}$ is a linear transformation such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

What are the possible dimensions of the null space $N(T)$? For each dimension, write down a possible form of T (i.e, give how T acts on a general vector in $\mathbb{R}^2$).