MAT 2377 - Practice Final Solutions

1. For the circuit in series, label the circuit as $E_8 = E_1 \cap E_2 \cap E_3$, which will function only if E_1 and E_2 and E_3 functions.

   For the second part, the circuit in parallel will operate only if E_4 or E_5 or E_6 operates, i.e. $E_9 = E_4 \cup E_5 \cup E_6$.

   Combinining with E_7 in series, the second half of the circuit operates if E_9 and E_7 operate, i.e. $E_{10} = E_9 \cap E_7$.

   So the first part will work with probability

   $$\begin{aligned} P(E_8) = P(E_1 \cap E_2 \cap E_3) &= P(E_1)P(E_2)P(E_3) \quad \text{by independence} \\ &= (0.99)^3 \\ &= 0.970299 \end{aligned}$$

   The second part operates with probability

   $$\begin{aligned} P(E_{10}) = P(E_9 \cap E_7) &= P(E_9)P(E_7) \quad \text{by independence} \\ &= [P(E_4 \cup E_5 \cup E_6)](0.95) \\ &= [1 - P(E_4')P(E_5')P(E_6')](0.95) \quad \text{using De Morgan's law} \\ &= [1 - (0.01)(0.03)(0.03)](0.95) \\ &= [0.999991](0.95) \\ &= 0.94999 \end{aligned}$$

   Then the entire circuit operates if

   $$\begin{aligned} P(E_8 \cup E_{10}) = 1 - P(E_8')P(E_{10}') &= 1 - (0.050008)(0.02701) \\ &= 0.9985 \end{aligned}$$

   $\rightarrow$ Answer (B)

2. Define events C - coarse edge
   D - depth above target

   We want to find $P(D|C) = \dfrac{P(D \cap C)}{P(C)} = \dfrac{15/200}{25/200} = \dfrac{15}{25}$

   $\rightarrow$ Answer (B).

3. We have that $P(A \cap B) = \frac{20}{200} = 0.1$

and $P(A) = \frac{45}{200}$, $P(B) = \frac{110}{200}$ so $P(A)P(B) = \left(\frac{45}{200}\right)\left(\frac{110}{200}\right)$
$= 0.12375 \neq 0.1$

So events A and B are dependent $\rightarrow$ Answer (B)

4. Since σ is unknown and n is small ($n=15$), we use the t-distribution, i.e.

$$T_0 = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{\nu = n-1}$$

Since $P(T_0 < c) = 0.05$ then $P(T_0 > c) = 0.95$ so with $\nu = n-1, = 14$ then note that $P(T_0 > -c) = 0.05$ also:

so $-c = 1.761 \Rightarrow c = -1.761$
$\rightarrow$ Answer (C)

5. We have $\hat{p} = \frac{18}{50} = 0.36$

Note $n\hat{p} = 50(0.36) = 18 > 5$ and $n(1-\hat{p}) = 50(0.64) = 32 > 5$ so the approximation is good.

A 95% confidence interval for p is ($\alpha = 0.05$, $\alpha/2 = 0.025$)

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.36 \pm (1.96)\sqrt{\frac{(0.36)(0.64)}{50}}$$

$$= 0.36 \pm 0.133$$
$$= [0.227, 0.493]$$

$\rightarrow$ Answer (A).

6. X = weight of packaged box

$X \sim \text{Normal}(\mu = 12.2,\ \sigma^2 = 0.0036)$ so $\sigma = \sqrt{0.0036} = 0.06$

$$P(X < 12) = P\left(\frac{X-\mu}{\sigma} < \frac{12-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{12-12.2}{0.06}\right)$$

$$= P(Z < -3.33)$$

$$= \Phi(-3.33)$$

$$= 0.0004 \rightarrow \text{Answer (B)}$$

7. Since σ is known, use a z-test. If the true mean $\mu = 3470$ then the sampling distribution of the z-test statistic is

$$Z_0 \sim N\left(\frac{\sqrt{n}(\mu_1 - \mu_0)}{\sigma}, 1\right) = N\left(\frac{\sqrt{25}(3470-3500)}{57}, 1\right)$$
$$= N(-2.63, 1)$$

Hence, the probability of committing a type II error if in reality $\mu = 3470$ is

$$\beta(3470) = P(\underline{\text{not}} \text{ falling in the critical region} \mid \mu = 3470)$$
$$= P(-z_{\alpha/2} < Z_0 < z_{\alpha/2} \mid \mu = 3470)$$

Since $z_{.025} = 1.96$ $= \Phi\left(\frac{1.96-(-2.63)}{\sqrt{1}}\right) - \Phi\left(\frac{-1.96-(-2.63)}{\sqrt{1}}\right)$

$$= \Phi(4.59) - \Phi(0.67)$$

$$\approx 1 - 0.748571$$

$$= 0.2514 \rightarrow \text{Answer (B).}$$

8. For a two-sided z-test,

p-value = P(reject Ho when Ho is true) = $2(1-\Phi(|z_0|))$

The test statistic is

$$Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} = \frac{3465-3500}{57/\sqrt{25}} = -3.07$$

So the p-value is $= 2(1-\Phi(|-3.07|))$
$= 2(1-0.998930)$
$= 0.0021$ → Answer (A)

9. The point estimate of μ is $\bar{x} = 323.24$ with estimated standard error

$$\frac{s}{\sqrt{n}} = \frac{9.991482}{\sqrt{15}} = 2.579790$$

→ Answer (C)

10. Define events
A - supplier A P(A) = 0.3
B - supplier B P(B) = 0.6
C - supplier C P(C) = 0.1

and event E - performs to specifications
$P(E|A) = 0.95$, $P(E|B) = 0.8$, $P(E|C) = 0.65$.

We want $P(A|E)$, so using Bayes theorem,

$$P(A|E) = \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C)}$$

$$= \frac{(0.95)(0.3)}{(0.95)(0.3) + (0.8)(0.6) + (0.65)(0.1)}$$

$$= \frac{0.285}{0.83}$$

$= 0.3434$ → Answer (E)

11. With 95% confidence, $z_{\alpha/2} = z_{.025} = 1.96$. Also $E = 15$ and $\sigma = 40$, so the sample size to compute the sample mean is

$$n \geq \left(\frac{z_{\alpha/2}\,\sigma}{E}\right)^2 = \left(\frac{1.96(40)}{15}\right)^2 = 27.3 \approx \boxed{28}$$

→ Answer (C)

12. For a continuous random variable X,

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu_X^2$$
$$= \int_0^1 x^2(3x^2)\,dx - (0.75)^2$$
$$= \int_0^1 3x^4\,dx - (0.75)^2$$
$$= \left.\frac{3x^5}{5}\right|_0^1 - (0.75)^2$$
$$= (3/5 - 0) - 0.5625$$
$$= 0.0375$$

So then the standard deviation is $\sigma_X = \sqrt{0.0375} = \boxed{0.1936}$

→ Answer (B)

13. Define X = no. of mice inoculated until the first mice contracts the disease

$X \sim \text{Geometric}\,(p = 1/6)$

So $P(X=8) = (1-p)^{x-1}p = \left(1-\frac{1}{6}\right)^7\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right) = \boxed{0.0465}$

→ Answer (A)

14. We have $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n} = \dfrac{25}{12} = 2.08333$ $\quad \bar{y} = \dfrac{\sum_{i=1}^{n} y_i}{n} = \dfrac{432}{12} = 36$

and $S_{xy} = \sum_{i=1}^{12} x_i y_i - n\bar{x}\bar{y} = \sum_{i=1}^{n} x_i y_i - \left(\dfrac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n}\right)$

$$= 880.5 - \frac{(25)(432)}{12} = -19.5$$

$$S_{xx} = \sum_{i=1}^{12} x_i^2 - \frac{(\sum x_i)^2}{n} = 59 - \frac{(25)^2}{12} = 6.91666$$

and $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} = \dfrac{-19.5}{6.91666} = -2.819277$

with $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = 36 - (-2.819277)(2.08333) = 41.87349$

Then $\hat{\mu}_{Y|x=5} = \hat{\beta}_0 + \hat{\beta}_1 x = 41.87349 + (-2.819277)(5)$
$= \boxed{27.78} \rightarrow$ Answer (A)

15. We have $\hat{\sigma}^2 = \dfrac{SSE}{n-2} = \dfrac{S_{yy} - \hat{\beta}_1 S_{xy}}{n-2} = \dfrac{4.1289 - (0.83075)(4.4094)}{14-2}$
$= 0.0388159$

Since $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} \Rightarrow S_{xx} = \dfrac{S_{xy}}{\hat{\beta}_1} = \dfrac{4.4094}{0.83075} = 5.3077339$

So the estimated standard error of $\hat{\beta}_1$ is

$$\hat{\sigma}_{\hat{\beta}_1} = \sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} = \sqrt{\frac{0.0388159}{5.3077339}} = \boxed{0.0855} \rightarrow \text{Answer (C)}$$