# University of Ottawa

# MAT 1332B Midterm Exam – Solutions

**March 25, 2009. Duration: 80 minutes. Instructor: Frithjof Lutscher**

**Question 1.** [12 points] Consider the following matrix:

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}.$$

1. Calculate the determinant and explain why the matrix is invertible. (One short sentence is enough.)
2. Find $A^{-1}$.
3. Solve the equation $Ax = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$.
4. Show that $\lambda_1 = -4$ is an eigenvalues of $A$ and find the other two eigenvalues.
5. Find the eigenvectors corresponding to $\lambda_1 = -4$.

**Solution**

$$\det(A) = (-4) \cdot 2 \cdot 2 - 4 \cdot 4 \cdot (-4) = -16 + 64 = 48 > 0$$

Since the determinant is not zero, the matrix in invertible. To calculate $A^{-1}$ we solve

$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/6 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 & -1/6 \end{array}\right]$$

Next, we have $x = A^{-1}b$, so

$$x = \begin{bmatrix} -1/4 & 0 & 0 \\ 0 & -1/6 & 1/3 \\ 0 & 1/3 & -1/6 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1 \\ 1 \end{bmatrix}$$

The eigenvalues are given by the roots of $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = (-4 - \lambda)(2 - \lambda)(2 - \lambda) - 16(-4 - \lambda) = (-4 - \lambda)(\lambda^2 - 4\lambda - 12).$$

Hence, the eigenvalues are $\lambda_1 = -4, \lambda_2 = 6, \lambda_3 = -2$.

For the eigenvector corresponding to $\lambda = -4$ we have to solve the system

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 4 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

The first column has no leading entry, $v_1 = t$ is a free variable. The eigenvector is $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

**Second version**
Same determinant.

$$A^{-1} = \begin{bmatrix} -1/6 & 0 & 1/3 \\ 0 & -1/4 & 0 \\ 1/3 & 0 & -1/6 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix}$$

Eigenvalues same as above. Eigenvector for $\lambda = -4$ is $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

**Third version**
Same determinant.

$$A^{-1} = \begin{bmatrix} -1/6 & 1/3 & 0 \\ 1/3 & -1/6 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} 1 \\ 1 \\ -3/2 \end{bmatrix}$$

Eigenvalues same as above. Eigenvector for $\lambda = -4$ is $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

**Question 2.** [4 points] Consider the following function of two variables:

$$f(x,y) = 1 - \frac{2x}{y} + 3y - 4xy^2 + e^{3x}.$$

1. Find the partial derivatives of $f$ with respect to $x$ and $y$.
2. Find the linear approximation at the point $(x,y) = (0,1)$.

**Solution**

$$\frac{\partial f}{\partial x} = -\frac{2}{y} - 4y^2 + 3e^{3x}, \qquad \frac{\partial f}{\partial y} = \frac{2x}{y^2} + 3 - 8xy$$

Near $(0,1)$ we have the approximation

$$f(x,y) \approx -3x + 3y + 2$$

**Second version**

$$f(x,y) = 5 + \frac{3y}{x} - 2x + 4x^3y - e^{-2y}.$$

$$\frac{\partial f}{\partial x} = -\frac{3y}{x^2} - 2 + 12x^2y, \qquad \frac{\partial f}{\partial y} = \frac{3}{x} + 4x^3 + 2e^{-2y}$$

Near $(1,0)$ we have the approximation

$$f(x,y) \approx -2x + 9y + 4$$

**Third version**

$$f(x,y) = -2 - \frac{9y}{x} + 3y - 5x - 3x^2y^3 - e^{2y}.$$

$$\frac{\partial f}{\partial x} = \frac{9y}{x^2} - 5 - 6xy^3, \qquad \frac{\partial f}{\partial y} = -\frac{9}{x} + 3 + 9x^2y^2 - 2e^{2y}$$

Near $(1,0)$ we have the approximation

$$f(x,y) \approx -5x - 8y - 3$$

**Question 3.** [5 points] Bobby the bird lives on Hawaii, where he travels between the islands of Maui (M) and Big Island (B). People tell you that Bobby's movement between M and B can be modeled as a Markov chain with the transition matrix

$$P = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix},$$

where 0.6 is the probability that Bobby will stay in M next week if he is there this week.

- Assume that Bobby is on M this week. What is the probability that he is on M in two weeks?
- What is the percentage of time that Bobby spends on M in the long run?

**Solution:**
Since we know that Bobby is in M this week, the vector is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then we compute $x_2 = P^2 x_0 = \begin{bmatrix} 0.56 \\ 0.44 \end{bmatrix}$. Therefore, the probability in two weeks is 56%.

The eigenvector of $p$ to the eigenvalue $\lambda = 1$ is $\begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$. Hence the probability for Bobby to be on M in the long run is 55.5%.

**Second version**
$x_2 = P^2 x_0 = \begin{bmatrix} 0.68 \\ 0.32 \end{bmatrix}$. Therefore the probability for Bobby to be in M in two weeks is 68%.

The eigenvector of $P$ to the eigenvalue $\lambda = 1$ is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Hence the probability for Bobby to be on M in the long run is 50%.

**Third version**
$x_2 = P^2 x_0 = \begin{bmatrix} 0.36 \\ 0.64 \end{bmatrix}$. Therefore the probability for Bobby to be in M in two weeks is 36%.

The eigenvector of $P$ to the eigenvalue $\lambda = 1$ is $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. Hence the probability for Bobby to be on M in the long run is 33.3%.

**Question 4.** [4 points] Consider the system of linear equations

$$\begin{aligned} x + ay &= 1 \\ bx + 5y &= 2 \end{aligned}$$

where $a$ and $b$ are parameters.

1. Determine the conditions on $a$ and $b$ to get a unique solution.
2. Determine the conditions on $a$ and $b$ to get infinitely many solutions.
3. Determine the conditions on $a$ and $b$ such that the system has no solutions.

**Solution**
If $b = 0$ then there is a unique solution for every value of $a$. If $b \neq 0$, then row reduction gives

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ b & 5 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 5 - ab & 2 - b \end{array}\right]$$

Hence, if $ab \neq 5$ then there is a unique solution. If $ab = 5$ and $b = 2$ (i.e., $a = 2.5$), then there are infinitely many solutions. If $b \neq 2$ and $ab = 5$, i.e. $a \neq 2.5$ then there is no solution.

**Second version:**
If $b = 0$ then there is a unique solution for every value of $a$. If $b \neq 0$, then row reduction as above. If $ab \neq 6$ then there is a unique solution. If $ab = 6$ and $b = 3$ (i.e., $a = 2.$), then there are infinitely many solutions. If $b \neq 3$ and $ab = 6$, i.e. $a \neq 2$ then there is no solution.

**Second version:**
If $b = 0$ then there is a unique solution for every value of $a$. If $b \neq 0$, then row reduction as above. If $ab \neq 7$ then there is a unique solution. If $ab = 7$ and $b = 4$ (i.e., $a = 1.75.$), then there are infinitely many solutions. If $b \neq 4$ and $ab = 7$, i.e. $a \neq 1.75$ then there is no solution.

**Question 5.** [5 points] Consider the equation $x^3 - 4x^2 - 2x + 20 = 0$

1. Show that $x_1 = -2$ is a solution of the equation.
2. Use long division to show that the other two roots are $x_2 = 3 + i$ and $x_3 = 3 - i$.
3. Calculate $x_2 x_3$.
4. Express $x_3$ in the form $x_3 = re^{i\theta}$.

**Solution**

The polynomials factors (long division) into

$$x^3 - 4x^2 - 2x + 20 = (x + 2)(x^2 - 6x + 10).$$

Applying the solution formula for quadratic equations to the second factor gives the roots $x_2 = 3 + i$ and $x_3 = 3 - i$.
The product is $x_2 x_3 = (3 + i)(3 - i) = 10$.
The absolute value is $|x_2| = \sqrt{10}$. The argument is $\theta = \tan^{-1}(1/3) = 0.3218$.

**Second version**

The polynomials factors (long division) into

$$x^3 - 5x^2 + 4x + 10 = (x + 1)(x^2 - 6x + 10).$$

The rest is as above.

**Third version**

The polynomials factors (long division) into

$$x^3 - 2x^2 - 14x + 40 = (x + 4)(x^2 - 6x + 10).$$

The rest is as above.