***Total mark: 40. Closed book. Non-programmable calculators are allowed.***

***Last Name*** ____________ ***First Name*** ____________ ***Student Number*** ____________

**Question** 1. [6 Points] Multiple Choice Questions(Please **CIRCLE** your answer )

(1) (2 points) Which of the following sets of vectors are linearly dependent?

$$A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} \right\} \quad B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 22 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 100 \\ 9 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix} 7 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

A. $A, C, D$ only **B.** $B, C, D$ **only** C. $A, B, D$ only D. $A, B, C$ only

(2) (2 points) Which of the following sets of vectors spans $\mathbb{R}^3$?

$$A = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

A. $A, D$ only **B.** $D, C$ **only** C. $C, B$ only D. $B, A$ only

(3) (2 points) Consider two matrices $A, B, C$. If $A$ is a $3 \times 4$ matrix, $B$ is a $4 \times 5$ matrix and $ABC$ is a $3 \times 6$ . Assume $ABC$ can be computed, what must be the dimensions of C

A. $3 \times 6$ B. $4 \times 4$ C. $5 \times 3$ **D.** $5 \times 6$

**Solution:** (1) B (2) B (3) D

**Question** 2. [10 Points] Let

$$A = \begin{pmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}, \; D' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

1) [3 points] Find $A - C'$.
2) [2 points]Find $B^4$.
3) [3 points]Find $BA$.
4) [2 points]Find $(C^T D)^T$.

**Solution:**

$$A - C' = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B^4 = \begin{pmatrix} 1 & 0 \\ -12 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & 4 & 3 \\ -14 & -11 & -8 \end{pmatrix}$$

$$(C^T D)' = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$$


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**Question** 3. [24 Points] Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

and the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{0.1}$$

($\mathbf{v}_i (i = 1, 2, 3)$ is the $i$th column of $A$). Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

1) [6 points] Solve the matrix equation $A\mathbf{x} = \mathbf{u}$.

**Solution:** The augmented matrix[1 points] is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 1 \end{array}\right]$$

(1 point for each correct row operation, maximum 3 points). The RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

or REF is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So no solution.

2) [2 points] Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent? Why or why not?

**Solution:** The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent[1 point], since the row rank is 2[1 point].

3) [2 points] Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ linearly independent? Why or why not?

**Solution:** The vectors $\mathbf{v}_1, \mathbf{v}_3$ are linearly independent[1 point], since the row rank is 2[1 point] .

4) [3 points] Determine if it is possible to express the vector $\mathbf{u}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. If it is possible, write $\mathbf{u}$ explicitly as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

**Solution:** No[1 point], since no solution for $A\mathbf{x} = \mathbf{u}$[2 points].

5) [2 points] Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}$ linearly independent or dependent? why ?

**Solution:** Dependent[1 point], since the rank is 3 or this set has 4 three dimensional vectors[1 point].

6) [3 points] Find the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ and write the solution in parametric vector form.

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**Solution:** The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbf{R}$$

7) [1 point] Is $\mathbf{u}$ in $W$, why?

**Solution:** No, since $A\mathbf{x} = \mathbf{u}$ is inconsistent.

8) [1 point] Is $\mathbf{v}_1$ in $W$?

**Solution:** Yes.

9) [2 points] Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$, why?

**Solution:** No. Since the row rank of A is 2.

10)[2 points] Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}\}$ span $\mathbb{R}^3$, why?

**Solution:** Yes. Since the row rank of the argument matrix in (1) is 3.