YOUR NAME

The University of Western Ontario
Calculus 1000 – Midterm Examination – Fall 2011
VERSION 111

Instructions

- Complete all thirty-one problems within the three-hour exam period.
- Write clearly in the space provided. Illegible solutions may receive a zero. If needed, you may continue your solution on the back of the previous page.
- **Show all your work** unless otherwise indicated. Incomplete answers may not receive full marks.
- Blank pages have been included at the end of the exam. These may be used for rough calculations. They will not be graded. **Do not detach blank pages** from the exam paper.
- Complete your solutions using **permanent ink unless otherwise indicated**. Marks received for answers given in something other than permanent ink will not be subject to appeal. **No correction fluid** and **no red ink**.
- The use of notes, books, calculators, communication devices, mp3 players, or computers during the examination is prohibited
- Print your name clearly at the top of every page in the space provided.
- Circle your section below.

Section	Instructor	Lectures
002	Khalkhali	NCB 113
003	Garrousian	3M 3250
004	MacIsaac	UCC 146
005	Kiriushcheva	SEB 2200
007	Harper	SSC 2050
008	Renner	NCB 113
009	Wild	SEB 2200
010	Moschandreou	MC 110
011	Harper	NCB 113
012	Xiao	SSC 2050
013	Shemyakova	UCC 146
014	DeGroot	SEB 1200
570	Bryan	King's College

YOUR NAME

Problems 1-25 are multiple choice. Enter the best answer on the ScanTron sheet using an HB or # 2 pencil. You do not need to show your work. In addition to entering your response on the ScanTron sheet, you may also wish to circle your response for your own records (ScanTron sheets will be graded, but will not be returned to you). **[2 mark each = 50 marks]**

1. Which (if any) of the following functions possesses an inverse?

   (A) $y(x) = 4$ (B) $y(x) = x^2$ (C) $y(x) = (x-3)x$ (D) $y(x) = x^4$ (E) none of (A) to (D)

2. If $f(-2) = 4$ and f is a one-to-one function, then we know that

   (A) $f^{-1}(2) = \frac{1}{4}$ (B) $f^{-1}(-2) = -\frac{1}{4}$ (C) $f^{-1}(-2) = 4$ (D) $f^{-1}(4) = -2$ (E) $f^{-1}(4) = -4$

3. If $f(x) = x^5 - 2x^4 + 2x^2 + x$, then $\lim_{x\to 1} f''(x) =$

   (A) -4 (B) -2 (C) 0 (D) 2 (E) 4

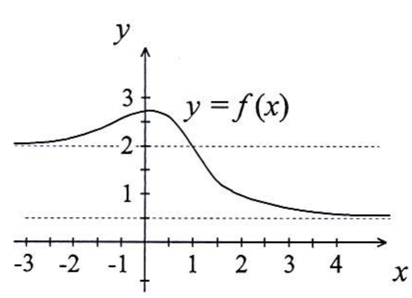
4. $\lim_{\theta\to 0} \frac{\sin\theta \cot\theta}{\cos\theta} =$

   (A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

5. $\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2} =$

   (A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

6. The graph of $y = f(x)$ is provided below. Based on the graph $\lim_{x\to -\infty} f(x) =$



   (A) 0

   (B) 1

   (C) 2

   (D) 3

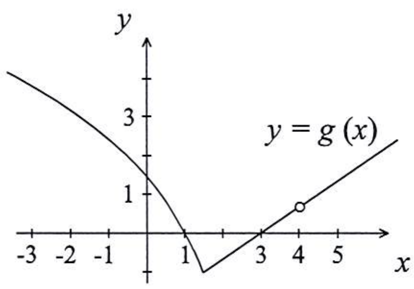
   (E) does not exist

YOUR NAME

7. $\lim_{t\to-\infty} \frac{4t^2-3}{7t+3} =$

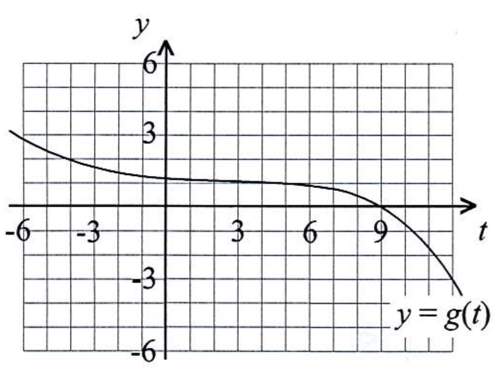
(A) $-\infty$ (B) $-\frac{4}{7}$ (C) 0 (D) $\frac{4}{7}$ (E) ∞

8. The graph of $y = g(x)$ is provided below. Based on the graph where is $\ln(g(x))$ continuous?



(A) $(0, 1)$, $(3, \infty)$
(B) $(-\infty, 1)$, $(3, 4)$, $(4, \infty)$
(C) $(0, 1)$, $(3, 4)$, $(4, \infty)$
(D) $(-\infty, 1)$, $(3, \infty)$
(E) $(0, \infty)$

9. The graph of $y = g(t)$ is provided below. Based on the graph one can say that $g'(9)$ is



(A) -2
(B) $-\frac{1}{3}$
(C) 0
(D) $\frac{1}{3}$
(E) not defined.

10. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) =$

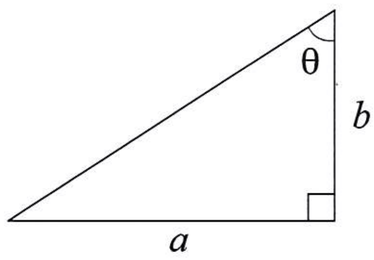
(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $-\frac{\sqrt{3}}{2}$ (E) none of (A) to (D)

11. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right) =$

(A) $\frac{\sqrt{3}}{2}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ (E) none of (A) to (D)

YOUR NAME

12. Based on the triangle below, $\tan^{-1}\frac{b}{a}$ is



(A) θ

(B) $\frac{\sqrt{3}}{2}$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}-\theta$

(E) $\frac{a}{\sqrt{a^2+b^2}}$

13. $\cos x \sin y + \cos y \sin x =$

(A) $\sin y \cos x$ (B) $\sin(x+y)$ (C) $\cos(x+y)$ (D) $\sin(2x)$ (E) none of (A) to (D)

14. If $F(\theta) = \cos\theta$, then $F^{(11)}(\theta) =$

(A) $\sin\theta$ (B) $\cos\theta$ (C) $-\sin\theta$ (D) $-\cos\theta$ (E) $-11\sin\theta$

15. $\lim_{x\to -1^+} \arcsin x =$

(A) $-\frac{\pi}{2}$ (B) -1 (C) $\frac{\pi}{2}$ (D) 1 (E) does not exist

16. $\frac{d}{dx}\arctan(e^x) =$

(A) $\frac{1}{1+x^2}e^x$ (B) $\frac{e^x}{\sqrt{1+e^{2x}}}$ (C) $\frac{e^x}{1+e^{2x}}$ (D) $\sec^2(e^x)$ (E) $e^x \sec^2(e^x)$

17. $\frac{d}{dx}\ln(\arcsin x) =$

(A) $\frac{1}{(\arcsin x)\sqrt{1-x^2}}$ (B) $\frac{-1}{(\arcsin x)\sqrt{1-x^2}}$ (C) $\frac{1}{(\arcsin x)\sqrt{1+x^2}}$

(D) $\frac{\arccos x}{\arcsin x}$ (E) $\frac{1}{\sqrt{1-(\arcsin x)^2}}$

YOUR NAME

18. If $y = 2^{\ln x}$, then $\left.\dfrac{dy}{dx}\right|_{x=e} =$

(A) $\dfrac{\ln 2}{2e}$ (B) $\dfrac{e\ln 2}{2}$ (C) $\dfrac{e}{2\ln 2}$ (D) $\dfrac{2\ln 2}{e}$ (E) 2

19. $\lim\limits_{x\to\frac{\pi}{2}^-} \arctan\left(\dfrac{1}{x-\frac{\pi}{2}}\right) =$

(A) $-\infty$ (B) ∞ (C) $-\dfrac{\pi}{2}$ (D) $\dfrac{\pi}{2}$ (E) 0

20. $\lim\limits_{t\to 0}\left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right) =$

(A) $-\infty$ (B) ∞ (C) 0 (D) -1 (E) 1

21. $\lim\limits_{x\to 0}\left(\dfrac{x}{\sqrt{1+3x}-1}\right) =$

(A) 0 (B) $\dfrac{2}{3}$ (C) 1 (D) 2 (E) ∞

22. If $f(x) = (3x+2)^3(2-x)^2$, then $f'(1) =$

(A) -25 (B) 150 (C) 450 (D) 475 (E) 0

23. If $g(x) = \ln(x^3-5)$ then $g^{-1}(x) =$

(A) $\sqrt[3]{e^x+5}$ (B) $\ln(\sqrt[3]{x}+5)$ (C) $\dfrac{1}{\ln(x^3-5)}$ (D) $e^{\sqrt[3]{x+5}}$ (E) $\sqrt[3]{e^x}-5$

24. $\lim\limits_{u\to\infty}\left(\dfrac{5u^2-1000u}{100\,000-u^2}\right) =$

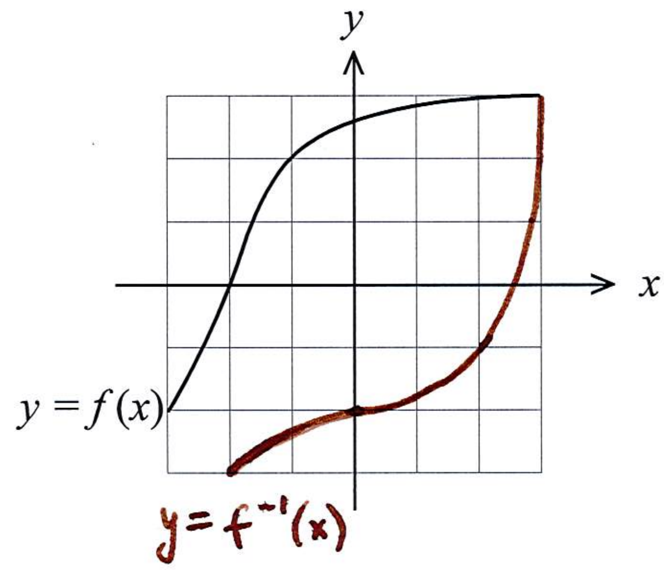
(A) 1 (B) -1 (C) 5 (D) -5 (E) does not exist

25. If $y(x) = x^x$, then $y'(x) =$

(A) x^{x-1} (B) x^x (C) $x^x(\pi - \cos x)$ (D) $x^x(1+\ln x)$ (E) none of (A) to (D)

YOUR NAME

26. (a) Given the graph of $y = f(x)$ below, plot $y = f^{-1}(x)$ on the same set of axes. **[3 marks]**



(b) Use the axes below to make one sketch of the graph of a function $y = f(x)$ for which each of the following is true:

(i) $\lim_{x\to\infty} f(x) = -1$

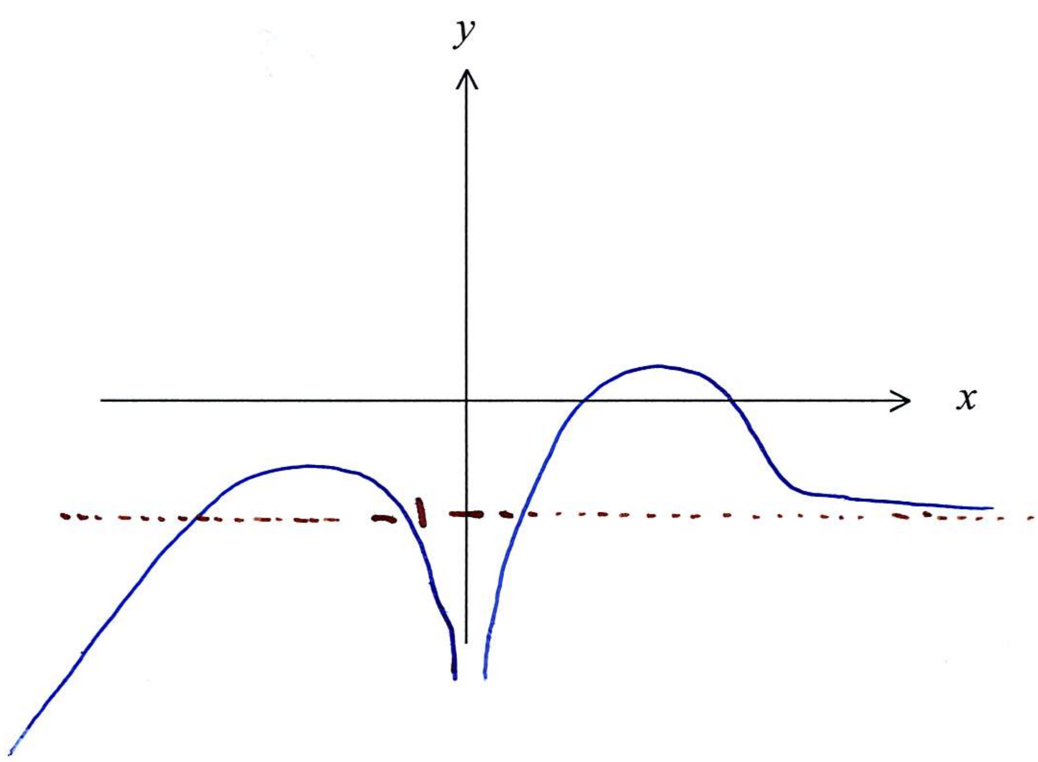
(ii) $\lim_{x\to-\infty} f(x) = -\infty$

(iii) $\lim_{x\to 0^+} f(x) = -\infty$

(iv) $\lim_{x\to 0^-} f(x) = -\infty$

(v) $f(x)$ has two, and only two, positive roots.

[5 marks]



YOUR NAME

27. Evaluate the given limit. If the limit does not exist, explain why.

(a) $\lim_{u \to -3} \frac{u^2 + 2u - 3}{|u + 3|}$ **[5 marks]**

$$f(u) = \begin{cases} \frac{u^2+2u-3}{-(u+3)} & u < -3 \\ \frac{u^2+2u-3}{u+3} & u > -3 \end{cases}$$

$$\lim_{u \to -3^-} f(u) = \lim_{u \to -3^-} \frac{u^2+2u-3}{-(u+3)} = \lim_{u \to -3^-} \frac{(u+3)(u-1)}{-(u+3)}$$

$$= \lim_{u \to -3^-} \left(-(u-1)\right) = 4$$

$$\lim_{u \to -3^+} f(u) = \lim_{u \to -3^+} \frac{u^2+2u-3}{u+3} = \lim_{u \to -3^+} \frac{(u+3)(u-1)}{(u+3)}$$

$$= \lim_{u \to -3^+} (u-1) = -4$$

Since one sided limits do not agree the limit, $\lim_{u \to -3} f(u)$ DNE.

(b) $\lim_{x \to 0} x^4 \sin \frac{1}{x}$ **[5 marks]**

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-x^4 \leq x^4 \sin \frac{1}{x} \leq x^4$$

$$\lim_{x \to 0} (-x^4) = \lim_{x \to 0} (x^4) = 0$$

∴ BY THE SQUEEZE THEOREM we can say that $\lim_{x \to 0} x^4 \sin \frac{1}{x} = 0$

YOUR NAME

28. Show that there exists a solution of the equation $\sqrt[3]{x} = 1 - x$ on the interval $(0, 1)$. **[5 marks]**

Consider $f(x) = \sqrt[3]{x} + x - 1$. This is continuous on $[0,1]$. In addition,

$f(0) = -1$ and $f(1) = 1$.

The number $N=0$ is between $f(0)$ and $f(1)$. therefore THE INTERMEDIATE VALUE THEOREM tells us that there exists some c, such that $0 < c < 1$ for which $f(c) = 0$

$$\Leftrightarrow \sqrt[3]{c} + c - 1 = 0$$

$$\Leftrightarrow \sqrt[3]{c} = 1 - c$$

$\therefore$ c is the solution we are trying to show exists.

YOUR NAME

29. Find an equation for the line tangent to the curve defined by $y\sin(2x) = x\cos(2y)$ at the point $(\frac{\pi}{2}, \frac{\pi}{4})$. **[7 marks]**

$$\frac{d}{dx}\left(y \sin 2x\right) = \frac{d}{dx}\left(x \cos 2y\right)$$

$$\frac{dy}{dx}\sin 2x + y\frac{d}{dx}(\sin 2x) = \cos 2y + x\frac{d}{dx}(\cos 2y)$$

$$\frac{dy}{dx}\sin 2x + 2y\cos 2x = \cos 2y - 2x\sin 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\sin 2x + 2x\sin 2y\right) = \cos 2y - 2y\cos 2x$$

$$\therefore \frac{dy}{dx} = \frac{\cos 2y - 2y\cos 2x}{\sin 2x + 2x\sin 2y}$$

and

$$\left.\frac{dy}{dx}\right|_{\substack{x=\pi/2 \\ y=\pi/4}} = \frac{\cos\frac{\pi}{2} - \frac{\pi}{2}\cos\pi}{\sin\pi + \pi\sin\frac{\pi}{2}} = \frac{1}{2}$$

If (x,y) is some point on the tangent line, then

$$\frac{y - \frac{\pi}{4}}{x - \frac{\pi}{2}} = \frac{1}{2}$$

$$y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

$$y - \frac{\pi}{4} = \frac{1}{2}x - \frac{\pi}{4}$$

$$\boxed{y = \frac{1}{2}x}$$

YOUR NAME

30. Use the **limit definition of a derivative** to solve parts (a) and (b).

(a) If $g(x) = \frac{1}{\sqrt{x}}$, then determine $g'(x)$. **[5 marks]**

$$g'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)}$$

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} \cdot \frac{1}{x(x+h)}$$

$$= -\frac{1}{2/\sqrt{x}} \cdot \frac{1}{x^2}$$

$$= -\frac{1}{2} \frac{1}{x^{3/2}}$$

$$= -\frac{1}{2} x^{-3/2}$$

(b) If $F(x) = \ln(x)$, then determine $F'(1)$. *Hint:* What is the definition of e? **[5 marks]**

$$F'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \ln(1+h)$$

$$= \lim_{h\to 0} \left(\ln\left((1+h)^{1/h}\right) \right)$$

$$= \ln\left(\lim_{h\to 0} \left((1+h)^{1/h}\right) \right)$$

$$= \ln(e) = 1$$

L'Hopital's Rule cannot be used, because applying it in this case assumes the conclusion.

YOUR NAME

31. Under certain circumstances a rumour spreads according to the equation $p(t) = \frac{1}{1+2e^{-t}}$, where $p(t)$ is the proportion of the population that knows of the rumour at time t.

(a) What fraction of the population knows of the rumour initially? **[2 marks]**

$$p(0) = \frac{1}{1+2e^0} = \frac{1}{3}$$

(b) What fraction of the population eventually knows of the rumour, in the long-run? **[2 marks]**

$$\lim_{t\to\infty} p(t) = \frac{\lim_{t\to\infty} 1}{\lim_{t\to\infty}\left(1+2e^{-t}\right)} = 1$$

(c) At what rate does the rumour spread at the instant half the population knows of it? **[6 marks]**

First, solve for time at which half the population knows:

$$\frac{1}{2} = \frac{1}{1+2e^{-t}}$$

$$2 = 1+2e^{-t}$$

$$1 = 2e^{-t}$$

$$\frac{1}{2} = e^{-t}$$

$$2 = e^{t}$$

$$\therefore t = \ln 2$$

Now,

$$p'(t) = \frac{d}{dt}\left(\left(1+2e^{-t}\right)^{-1}\right)$$

$$= -\left(1+2e^{-t}\right)^{-2}\left(-2e^{-t}\right)$$

$$= 2\left(1+2e^{-t}\right)^{-2}e^{-t}$$

So,

$$p'(\ln 2) = 2\left(1+2e^{-t}\right)^{-2}e^{-t}\Big|_{t=\ln 2}$$

$$= 2\left(1+2e^{-\ln 2}\right)^{-2}e^{-\ln 2}$$

$$= 2\left(1+2\cdot\tfrac{1}{2}\right)^{-2}\tfrac{1}{2}$$

$$= 1\cdot(2)^{-2}$$

$$\boxed{= \frac{1}{4}}$$