The University of British Columbia

10 October 2013

Common Midterm for All Sections of MATH 104 and 184

Closed book examination | Time: 60 minutes

Last Name SOLUTIONS **First** ________________

Signature ________________

Student Number ________________

MATH 104 or MATH 184 (Circle one) **Section Number:**________________

Special Instructions:

No memory aids are allowed. No calculators. No communication devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		18
2		8
3		10
4		7
5		7
Total		50

[21] 1. **Short Answer Questions**. Each question is worth 3 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$.

$$= \lim_{x\to 0} \frac{x^2+\cancel{x}-\cancel{x}}{x(x^2+x)}$$

1 mark for adding fractions

$$= \lim_{x\to 0} \frac{1}{x+1} = 1$$

1 mark for simplification

Answer: 1

3

(b) Evaluate $\lim_{y\to 1}\frac{y-1}{y^2-4}$.

1 mark for saying why you can "plug in."

$\frac{y-1}{y^2-4}$ is continuous at $y=1$.

Answer: 0.

$$\text{So } \lim_{y\to 1}\frac{y-1}{y^2-4} = \frac{1-1}{1-4} = 0.$$

3

1 mark for "plugging in."

(c) Find the derivative of $f(x) = \frac{\sin(5x)}{x^2+1}$.

Answer:

$$f'(x) = \frac{(\sin(5x))'(x^2+1)-\sin(5x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{5\cos(5x)(x^2+1)-\sin(5x)(2x)}{(x^2+1)^2}$$

3

1 mark for knowing quotient rule.

1 mark for knowing chain rule.

(Full marks for an unsimplified answer as well.)

(d) Find $g'(z)$ if $g(z) = z(e^{z^3} + 1)$.

$$g'(z) = (z)'(e^{z^3}+1) + z(e^{z^3}+1)'$$

$$= 1\cdot(e^{z^3}+1) + z(e^{z^3}\cdot 3z^2).$$

$$= e^{z^3}+1 + 3z^3e^{z^3} = (3z^3+1)e^{z^3} + 1.$$ 3

Answer:

1 mark for knowing product rule.

1 mark for knowing chain rule.

(e) Let $f(x) = x\sin(x)$. Find the equation of the tangent line to $f(x)$ at $x = \pi/4$.

Answer:

ACCEPTABLE FORM.

$$x = \frac{\pi}{4} \Rightarrow y = f(\tfrac{\pi}{4}) = \frac{\pi}{4}\sin(\tfrac{\pi}{4})$$

$$= \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}.$$

POSSIBLE 1 mark for correct y-value

$$f'(x) = \sin x + x\cos x$$

1 mark for correct derivative

$$\Rightarrow f'(\pi/4) = \sin(\pi/4) + \pi/4\cos(\pi/4) = \frac{1}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{4}\right)$$

1 mark for correct slope

Eqn of tangent line:

$$\boxed{y - \frac{\pi}{4\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{4}\right)(x - \pi/4)}$$ 3

Any correct simplification accepted

(f) At how many points on the graph of $y = f(x) = \dfrac{3x-5}{2x+2}$ is the slope of the tangent line equal to 1?

Answer: 2. 3

Compute $f'(x) = \dfrac{(3x-5)'(2x+2) - (3x-5)(2x+2)'}{(2x+2)^2}$

$= \dfrac{3(2x+2) - (3x-5)2}{(2x+2)^2} = \dfrac{16}{(2x+2)^2}$

1 mark for derivative

$\Rightarrow f'(x) = \dfrac{4}{(x+1)^2}$.

Set $\dfrac{4}{(x+1)^2} = 1$ 1 mark for equation

$\Rightarrow 4 = (x+1)^2 \Rightarrow$ OR $x^2 + 2x - 3 = 0$ which has 2 distinct roots via either factoring or noting discriminant $\neq 0$.

$\Rightarrow 2 = \pm(x+1)$

$\Rightarrow x = 1$ or $x = -3$

1 mark for counting solutions

(A definition with $\lim_{h \to 0}$ is also acceptable, of course.)

Expect a sentence, but mark generously.

[8] **2. Definition of the Derivative.**

(a) [3] Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

The derivative of a function $f(x)$ at a point $x=a$ is $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$, $\boxed{\text{provided this limit exists}}$. N.B.

2 marks

1 mark

(b) [5] Use the definition of the derivative to compute the derivative of $f(x) = 4 - \sqrt{x+3}$ at $x = 6$. NO CREDIT will be given for using any other method.

$$f'(6) = \lim_{x \to 6} \frac{f(x) - f(6)}{x-6}$$

$$= \lim_{x \to 6} \frac{(4 - \sqrt{x+3}) - (4 - \sqrt{6+3})}{x-6}$$

1 mark

$$= \lim_{x \to 6} \frac{4 - \sqrt{x+3} - 1}{x-6}$$

$$= \lim_{x \to 6} \frac{(3 - \sqrt{x+3})}{(x-6)} \cdot \frac{(3 + \sqrt{x+3})}{(3 + \sqrt{x+3})}$$

2 marks

$$= \lim_{x \to 6} \frac{9 - (x+3)}{(x-6)(3+\sqrt{x+3})}$$

1 mark

$$= \lim_{x \to 6} \frac{\cancel{6-x}^{\,-1}}{\cancel{(x-6)}(3+\sqrt{x+3})} = \boxed{\frac{-1}{6}}$$

1 mark

1 mark set-up

2 marks for the radical conjugate trick.

1 mark for algebra

1 mark for final answer.

[10] **3**. M-Wave Co. makes and sells the world's first pocket quantum computers! When each quantum computer is sold for \$500, the weekly demand is 4,000 units. For every \$1 increase in the price of each unit, the number of quantum computers sold per week decreases by 10. Assume that it costs \$300 to produce each quantum computer.

(a) [2] Find the linear demand equation for the M-Wave quantum computer. Use p for the unit price and q for the weekly demand.

Given $(q_0, p_0) = (4000, \$500)$ and slope $= \frac{\Delta p}{\Delta q} = \frac{\$1}{-10} = m$

1 mark for correct slope

We get $p - p_0 = m(q - q_0)$

$\Rightarrow p - 500 = -\frac{1}{10}(q - 4000)$

$\Rightarrow p = -\frac{1}{10} q + 900$

1 mark for correct eqn

(b) [1] Find the weekly cost function $C(q)$ *as a function of* q.

There are no given fixed costs, so

$C(q) = 300q.$

1 mark for correct formula

(c) [1] Find the weekly revenue function $R(q)$ *as a function of* q.

$R(q) = p \cdot q = p(q) \cdot q = \left(-\frac{1}{10} q + 900\right) q$

$\Rightarrow R(q) = -\frac{1}{10} q^2 + 900q.$

1 mark for any correct form of $R(q)$

(d) [2] Find the break-even points for the M-Wave quantum computer. Give both the price p and quantity q at each of these points.

Pts

(0, 900)

(6000, 300)

Break even $R(q) = C(q)$

$$\Rightarrow \boxed{-\frac{1}{10}q^2 + 900q = 300q}$$ 1 mark

$$\Rightarrow -\frac{1}{10}q^2 + 600q = 0$$

$$\Rightarrow q = 0 \text{ OR } q = 6000.$$ 1 mark

(e) [2] Find the marginal profit function $MP(q)$.

$$P(q) = \text{Profit} = R(q) - C(q) = -\frac{1}{10}q^2 + 900q - 300q$$

$$= -\frac{1}{10}q^2 + 600q$$

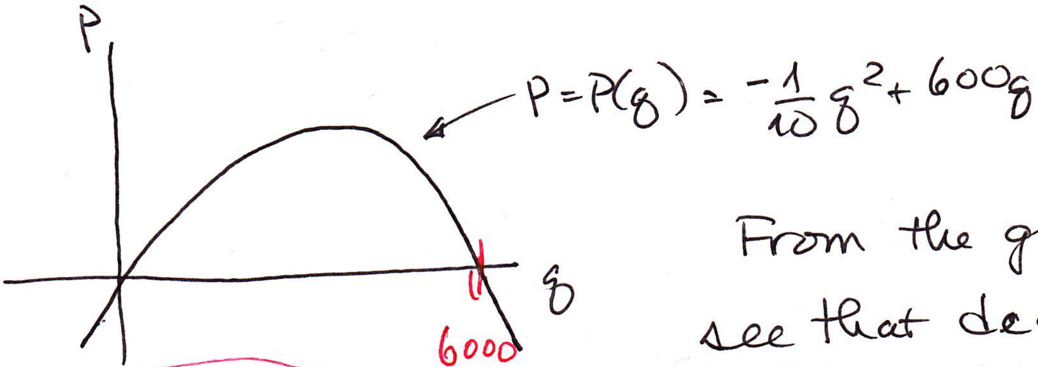
$$\Rightarrow MP(q) = \boxed{P'(q) = -\frac{1}{5}q + 600}$$ 1 mark

(f) [2] Suppose that M-Wave is producing and selling $\hat{q}$ quantum computers, where $\hat{q}$ corresponds to the largest q-value of all the break-even points. Should M-Wave increase or decrease the price of its quantum computers to increase its profit? Explain your answer.

$$\hat{q} = 6000. \quad MP(6000) = -\frac{6000}{5} + 600 = -600 < 0$$ 1

$\Rightarrow$ decrease q ~~So~~ increase p. 1

OR



1 mark for graph

From the graph, we see that decreasing q increases profit.

1 mark for conclusion

Students are unlikely to write much here. Want to see calculations using limits.

[7] 4. Let a and b denote constants. Find the values of a and b so that the function

$$f(x) = \begin{cases} 1+e^{-x} & \text{if } x<0, \\ ax+b & \text{if } x \geq 0 \end{cases}$$

is differentiable everywhere.

① Each piece of this function is differentiable on its part of the domain away from the point $x=0$.

1 mark

② Differentiability $\Rightarrow$ continuity, so to be differentiable at $x=0$, f must be continuous there:

1 mark for this idea.

That is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = b$.

1 mark for setting up calculation

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1+e^{-x} = \boxed{2 = b}$

1 mark answer.

③ To be ~~cont~~ differentiable at $x=0$, f must also have a tangent line at $x=0$, and so we must have that the "derivatives" of each component function match at $x=0$:

1 mark for idea

Ⓐ $(1+e^{-x})'\Big|_{x=0^-} = -e^{-x}\Big|_{x=0^-} = -1$.

1 mark for calculation set up.

Ⓑ $(ax+2)'\Big|_{x=0^+} = a$

$\Rightarrow \boxed{a=-1}$

1 mark answer.

[7] **5** Does $f(x) = x^3 - 1 + \sin(x)$ have a root in $(0, 1)$? Explain your answer carefully, and be sure to justify each assertion you make.

① $f(x)$ is a continuous function everywhere, and hence certainly continuous on $[0,1]$. 1 mark

② We can apply the Intermediate Value Theorem if we can find values of $f(x)$ on $[0,1]$ where $f(x)$ is positive and where it is negative, respectively. 2 marks

③ $f(0) = 0^3 - 1 + \sin(0) = -1 < 0$. 1 mark

④ $f(1) = 1^3 - 1 + \sin(1) = \sin(1) > 0$ + 1 mark
since $\sin(x) > 0$ on $(0, \pi/2)$
1 mark.

⑤ Hence there is a root of f ~~between~~ in the interval $(0,1)$ by the IVT. 1 mark

1 mark for continuity
2 marks for IVT.
1 mark for $f(0) < 0$
2 marks for $f(1) > 0$
1 mark conclusion.