MAT 1341E Assignment 2, 2011

Due: 6pm, 2-November, 2011.
Instructor: Sanghoon Baek

Family Name:______________________

First Name:________________________

Student number:____________________

(For the marker's use only →)

1	
2	
3	
4	
5	
6	
7	
[Bonus] 8	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 6 to 8, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
2. Questions 1 to 5 are worth 1 point each, and no part marks will be given. However, you must show some work that supports your answer to obtain the point. Simply writing the correct answer will earn 0.
3. Questions 6 and 7 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.** Question 8 is a bonus question and is worth 4 points. Earning points here will be much more difficult than in questions 1-7.
4. You may submit this assignment to me after class, or by putting into the box marked **"MAT1341 Section E"** in the cabinet in the foyer of the math building (KED) on the due date, by 6pm.

1. (1 mark) Consider a homogeneous system of 22 linear equations in 16 unknowns (variables). Let A be the augmented matrix of this system. If the RREF of A has 5 leading 1's (i.e., $\text{rank}(A) = 5$), how many parameters (variables) are there in the general solution?

: There are 16 variables. As the leading 1's corresponds to the leading variables, we have $16 - 5 = 11$ free variables in the general solution.

ANSWER | 11

2. (1 mark) Find the (2, 3)-entry of the product

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 5a & 1 \\ 4 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 2b & 3 & 2 \\ 5 & 1 & 0 \\ 0 & 4c & 3 \end{bmatrix}$$

: The (2, 3)-entry is $0 \cdot 1 + 2 \cdot 2 + 5a \cdot 0 + 1 \cdot 3 = 7$.

ANSWER | 7

3. (1 mark) Consider a homogeneous system of 10 linear equations in 6 unknowns (variables). Let A be the augmented matrix of this system. Suppose that the system has the only trivial solution $x_1 = x_2 = \cdots = x_6 = 0$.

- How many leading 1's are in the RREF of A?
- Do the columns of A span $\mathbf{R}^{10}$?

A. 0, Yes

B. 6, Yes

C. 6, No

D. 4, Yes

E. 4, No

F. 10, Yes

: As the system has a unique solution, the number of variables($= 6$) should be equal to the number of leading 1's. As $\dim(\mathbf{R}^{10}) = 10$, the 6 columns cannot span $\mathbf{R}^{10}$.

ANSWER C

4. (1 mark) Let $M_{2,2}(\mathbf{R})$ denote the vector space of real 2×2 matrices and let $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the dimension of the subspace $\{A \in M_{2,2}(\mathbf{R}) \mid KA = AK\}$.

: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

we have

$$KA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

and

$$AK = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

As $KA = AK$, we have $b = c$ and $a = d$. Hence,

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the dimension is 2.

ANSWER 2

5. (1 mark) The set of solutions S of the following system is a subspace of $\mathbf{R}^4$:

$$\begin{cases} u - 2x + 3y + 4z = 0 \\ -2u + 4x - 5y - 6z = 0 \end{cases}$$

- Find the set of solutions S of the system above.
- Find a basis for S.

: We have

$$\begin{bmatrix} 1 & -2 & 3 & 4 & 0 \\ -2 & 4 & -5 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \text{(REF)} \to \begin{bmatrix} 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \text{(RREF)}.$$

Let $x = s$ and $z = t$. Then $u = 2s + 2t$ and $y = -2t$. Hence, the set of solution is

$$(u, x, y, z) = (2s + 2t, s, -2t, t) = s(2, 1, 0, 0) + t(2, 0, -2, 1).$$

Therefore, $\{(2, 1, 0, 0), (2, 0, -2, 1)\}$ is a basis.

ANSWER

$(u, x, y, z) = (2s + 2t, s, -2t, t) = s(2, 1, 0, 0) + t(2, 0, -2, 1)$, $\{(2, 1, 0, 0), (2, 0, -2, 1)\}$.

6. (6 marks) Suppose $a, c \in \mathbf{R}$ and consider the linear system in variables x, y and z:

$$\begin{array}{rcrcrcr} x & & & + & z & = & -1 \\ 2x & + & y & + & z & = & -1 \\ x & - & 2y & + & az & = & c \end{array}$$

(a) If $[A \mid b]$ is the augmented matrix of the system above, find the number of leading 1's in the RREF of $[A \mid b]$ (i.e., rank$[A \mid b]$) and the number of leading 1's in the RREF of A (i.e., rank A) for all values of a and c.
Solution:

First of all, note that the number of leading 1's in a REF is equal to the number of leading 1's in the RREF when we apply Gaussian elimination. A REF of $[A \mid b]$ is

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a-3 & c+3 \end{bmatrix}.$$

Hence, the number of leading 1's of A is equal to

$$\begin{cases} 3 & \text{if } a \neq 3, \\ 2 & \text{if } a = 3. \end{cases}$$

and the number of leading 1's of $[A \mid b]$ is equal to

$$\begin{cases} 3 & \text{if } a \neq 3 \text{ or } c \neq -3, \\ 2 & \text{if } a = 3 \text{ and } c = -3. \end{cases}$$

(b) Using part (a), find all values of a and c so that this system has

(i) a unique solution,

a unique solution
$\Leftrightarrow$ the number of leading 1's of A = the number of leading 1's of $[A \mid b]$ = the number of variables (= 3)
$\Leftrightarrow a \neq 3$.

(ii) infinitely many solutions, or

infinitely many solutions
$\Leftrightarrow$ the number of leading 1's of A = the number of leading 1's of $[A \mid b]$ < the number of variables (= 3)
$\Leftrightarrow a = 3$ and $c = -3$.

(iii) no solutions.

no solution
$\Leftrightarrow$ the number of leading 1's of A < the number of leading 1's of $[A \mid b]$
$\Leftrightarrow a = 3$ and $c \neq -3$.

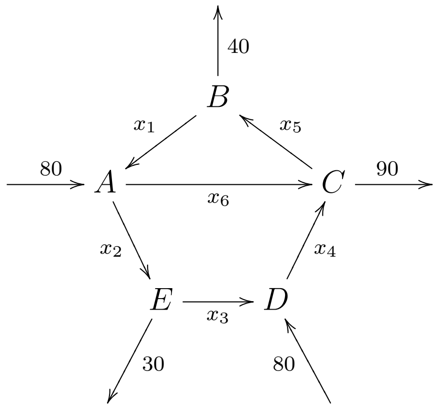
(c) In case b(ii) above, give a geometric description of the set of solutions.

If $a = 3$ and $c = -3$, then a REF of $[A \mid b]$ is

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, by setting $z = t$, we have $x = -t - 1$ and $y = t + 1$. Therefore, the set of solution is $(x, y, z) = (-1-t, t+1, t) = (-1, 1, 0) + t(-1, 1, 1)$, which is a line through $(-1, 1, 0)$ with the directional vector $(-1, 1, 1)$.

7. (6 marks) Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 6$.

(Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

A: $x_1 + 80 = x_2 + x_6$,
B: $x_5 = x_1 + 40$,
C: $x_4 + x_6 = x_5 + 90$,
D: $x_3 + 80 = x_4$,
E: $x_2 = x_3 + 30$,
where $x_1, x_2, x_3, x_4, x_5, x_6$ are all positive integers.

(b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

: Let $x_5 = s$ and $x_6 = t$. Then the set of solution is

$(x_1, x_2, x_3, x_4, x_5, x_6) = (s - 40, s - t + 40, s - t + 10, s - t + 90, s, t)$.

(c) If $\overline{AC}$ were closed due to roadwork, find the minimum flow along $\overline{ED}$, **using your results from (b).**

(*You must justify all your answers.*)

: By assumption, $x_6 = 0$, i.e., $t = 0$. As $x_i \geq 0$ for all $1 \leq i \leq 6$, we have $s \geq 40$. Hence, the minimum flow along $x_3 = s + 10$ is 50 when $s = 40$.

8. [Bonus] (4 marks) If A is any 2×2 matrix, prove that the number of leading 1's in the RREF of A is equal to the number of leading 1's in the RREF of A^T, where A^T is the transpose of A.

Solution: First of all, note that the number of leading 1's in a REF is equal to the number of leading 1's in the RREF when we apply Gaussian elimination.

Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ for some } a, b, c, d \in \mathbf{R}.$$
Then
$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$
If $b = c$, there is nothing to prove. Hence, we may assume that $b \neq c$. We divide the proof in two cases:

Case 1: If $a \neq 0$. Then a REF of A is
$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix}$$
and a REF of A^T is
$$\begin{bmatrix} 1 & \frac{c}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix}$$
Hence, they have the same number of leading 1's.

Case 2: If $a = 0$. Then there are three subcases:

Case 2.1: If $c \neq 0$ and $b \neq 0$. Then a REF of A is
$$\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix}$$
and a REF of A^T is
$$\begin{bmatrix} 1 & \frac{d}{b} \\ 0 & c \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{b} \\ 0 & 1 \end{bmatrix}$$
Hence, they have the same number of leading 1's.

Case 2.2: If $c = 0$ and $b \neq 0$. Then a REF of A is
$$\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & d \end{bmatrix}$$
and a REF of A^T is
$$\begin{bmatrix} b & d \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{b} \\ 0 & 0 \end{bmatrix}$$
Hence, they have the same number of leading 1's.

Case 2.3: If $b = 0$ and $c \neq 0$. Then a REF of A is
$$\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 0 \end{bmatrix}$$
and a REF of A^T is
$$\begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & d \end{bmatrix}$$
Hence, they have the same number of leading 1's.