A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E 1-1

The set D is called the domain of f

The number (element) $f(x)$ is the value of f at x

The range of f is the set of all possible values of $f(x)$ as x varies throughout D

(range = image)

Independent variable = a symbol that represents an arbitrary number (element) in D

Dependent variable = a symbol that represents a number (element) in the range of f.

Examples: ① The area S of a square depends on the side a of the square. The rule that connects a and S is given by the equation 1-2

$S = a^2$.

With each positive number a there is one value of S and we say that S is a function of a

② The price P of house depends on the time t.

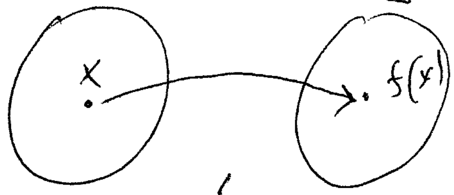
The table gives estimates of the cost at time t for certain years

Year	Cost
1995	120K
1996	122K
1997	128K
1998	118K

So for each value of time t there is a corresponding value of P.

$\Rightarrow P$ is a function of t.

One way to picture a function is by an arrow diagram (1-3)



$D \xrightarrow{f} E$ or $f : D \longrightarrow E$

The most common method for visualising a function is its graph.

If f is a function with domain D, then its graph is the set of ordered pairs

$$\{ (x, f(x)) \mid x \in D \}$$

So the graph consists of all points (x, y) such that $y = f(x)$ and $x \in D$.

Examples

The graph of a function is a curve in the xy-plane (1-4)

Which curves in the xy-plane are graphs of functions

Answer: (the vertical line test) A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Examples of right and wrong graphs.

Question: Sketch the graph and find the domain and range of the function (1-5)

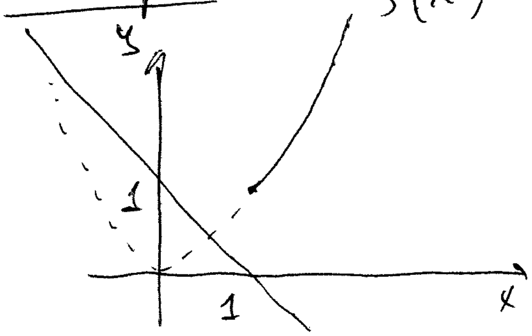
$$g(x) = |\sqrt{x}|.$$

5 minutes

Piecewise Defined Functions

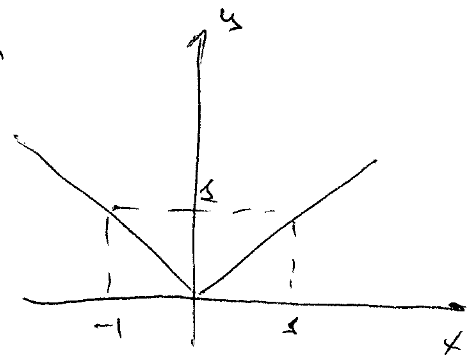
Example

$$f(x) = \begin{cases} 1 - x & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$



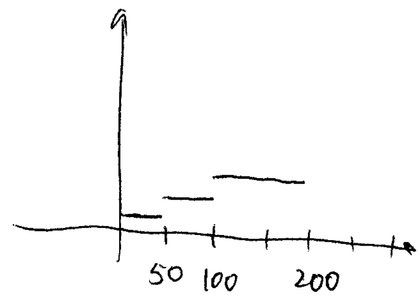
absolute value

$$f(x) = |x|$$



cost of stamp

$$C(w) = \begin{cases} 0.55 & \text{if } 0 < w \le \\ 1.22 & \text{if } 50 \le w \\ 2.44 & \text{if } 100 < w \le \end{cases}$$



Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an <u>even</u> function.

The graph of an even function is symmetric with respect to the y-axis

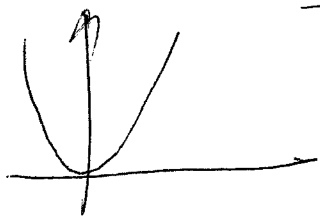
If $f(-x) = -f(x)$ for every $x \in D$, then f is called an <u>odd</u> function.

Its graph is symmetric about the origin.

Examples of even and odd func

Increasing and Decreasing functions (1-8)

A function f is called <u>increasing</u> on an interval I if
$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$
It is called decreasing on I if
$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$
<u>Examples of</u> increasing and decreasing functions.



<u>Polynomials</u> (1-9)

A function $P(x)$ is called a polynomial if $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ where n is a non-negative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the coefficients of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, +\infty)$

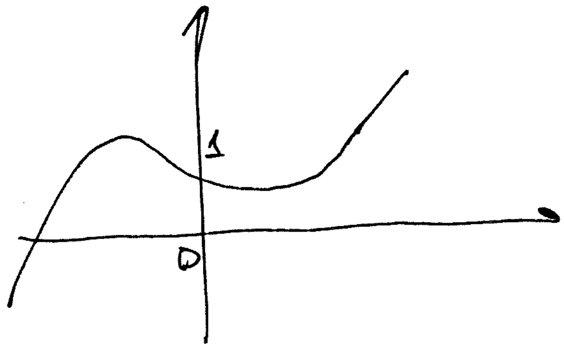
If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

A polynomial of degree 2
$$P(x) = ax^2 + bx + c \leftarrow \text{a quadratic function}$$

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \quad , \quad a \neq 0 .$$

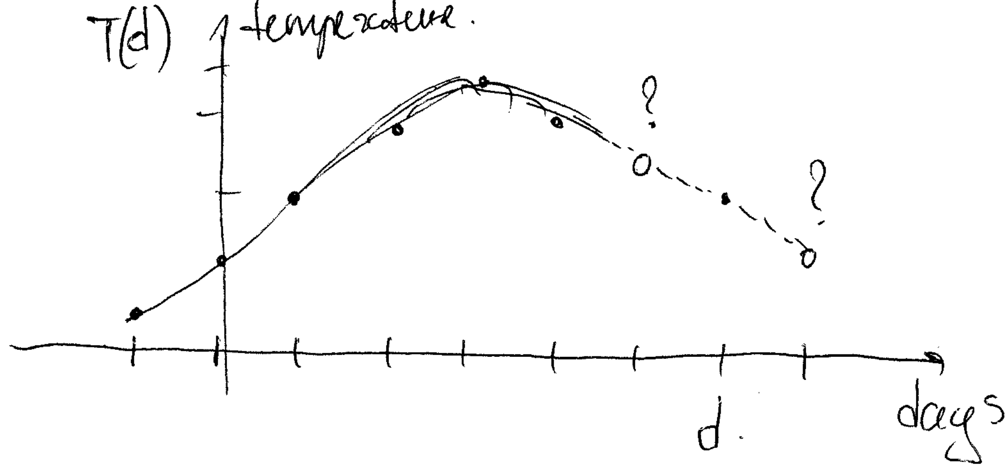
and is called a cubic fúnction.



$y = x^3 - x + 1$

A mathematical model is a mathematical description (by means of a function) of a real-world phenomenon. (1-1)

The purpose of the model is to understand the phenomenon and to make predictions about future behaviour



Linear model (1-2)

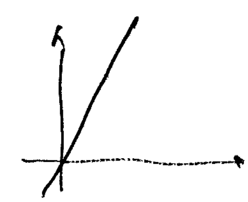
y is a linear function of x if its graph is a line, e.g.

$$y = f(x) = mx + b$$

m is the slope

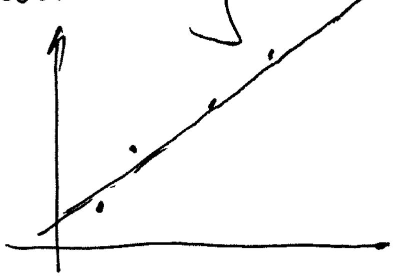
b is the y-intercept.

x	values
1	2
2	4
3	6



Linear functions grow at a constant rate.

A linear regression model



Example (a cubic model) 2-B

Your response Correct response

Determine the unique cubic polynomial (that is, polynomial of degree 3) f such that $f(0) = 0$, $f(1) = 10$, $f(-1) = -6$ and $f(2) = 54$.

$f(x) = 5*x^3 + 2x^2 + 3*x$

The most general form of a polynomial of degree 3 is

$$f(x) = ax^3 + bx^2 + cx + d$$

So suppose f is in this form; we'll solve for a, b, c and d.
Substituting $f(0) = 0$ gives:

$$0 = a(0)^3 + b(0)^2 + c(0) + d$$

whence $d = 0$. So f in fact has the simpler form

$$f(x) = ax^3 + bx^2 + cx.$$

Now we use the given equations: $f(1) = 10$, $f(-1) = -6$ and $f(2) = 54$, which give the following system of linear equations in a, b and c:

$$10 = a(1)^3 + b(1)^2 + c(1) = a + b + c$$

$$-6 = a(-1)^3 + b(-1)^2 + c(-1) = -a + b - c$$

$$54 = a(2)^3 + b(2)^2 + c(2) = -8a + 4b - 2c$$

We solve this as follows: adding the first two equations together gives an equation only in b, and we solve it to get $b = 2$. Then, adding 2 times the first equation to the third lets us solve for $a = 5$ and finally $c = 3$. Hence

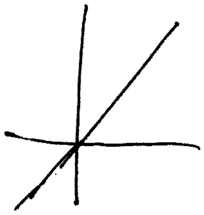
$$f(x) = 5x^3 + 2x^2 + 3x$$

Power functions

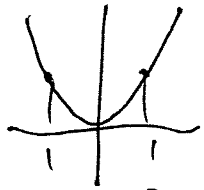
A function of the form $f(x)=x^a$, where a is a constant, is called a power function.

(i) $a=n$, n is a positive integer

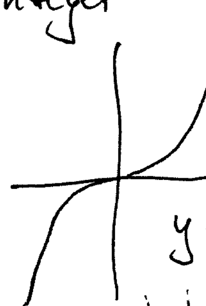
Graphs



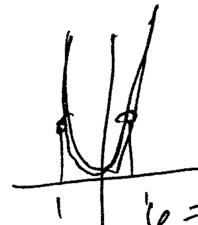
$y=x$



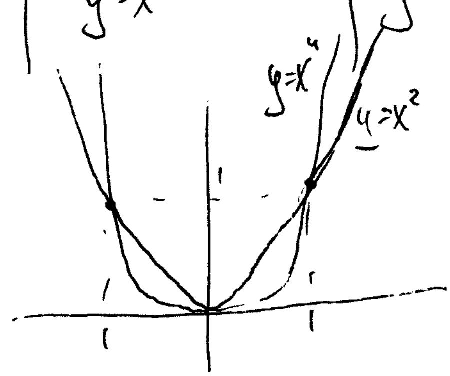
$y=x^2$



$y=x^3$



$y=x^4$



If n is even $f(x)=x^n$ is even

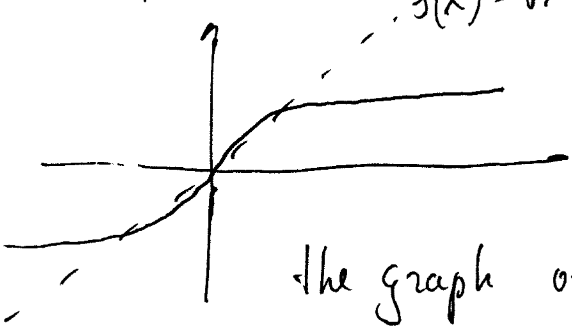
-"- odd -"- odd

(ii) $a=\frac{1}{n}$ n is a positive integer

The function $f(x)=x^{\frac{1}{n}}=\sqrt[n]{x}$ is a root function

$f(x)=\sqrt{x}$ Domain : $[0;+\infty)$

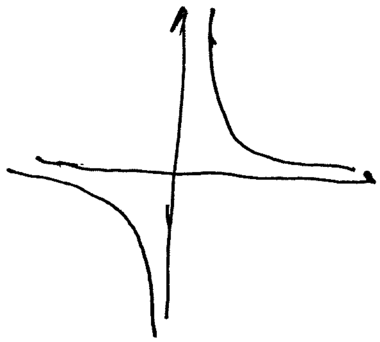
$f(x)=\sqrt[3]{x}$



~~turned~~ reflected $f(x)=x^3$.

the graph of $f(x)=\sqrt[4]{x}$ is similar to $\begin{cases} \sqrt{x} & \text{for even} \\ \sqrt[3]{x} & \text{for odd} \end{cases}$

(iii) $a = -1$ reciprocal function $f(x) = \frac{1}{x}$ 2-6



odd function

Rational functions

A rational function f is a ratio of two polynomials $f(x) = \frac{P(x)}{Q(x)}$ polynomials

Domain consists of all values of x s.t. $Q(x) \neq 0$.

Example $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$ Its domain is $\{x \mid x \neq \pm 2\}$.

Algebraic functions 2-7

A function is called algebraic if it can be constructed using algebraic operations (addition, multiplication, subtraction, division and takin roots) starting with polynomials.

Any rational function is an algebraic function.

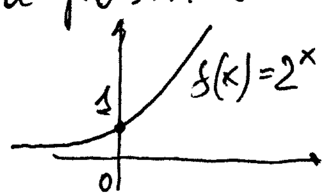
Ex. $f(x) = \sqrt{x^2+1}$

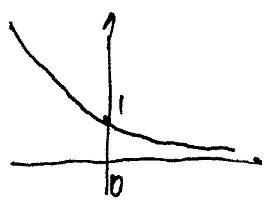
Exponential functions

are of the form $f(x) = a^x$, where the base a is a positive constant.

The domain is $(-\infty, +\infty)$

The range is $(0, +\infty)$



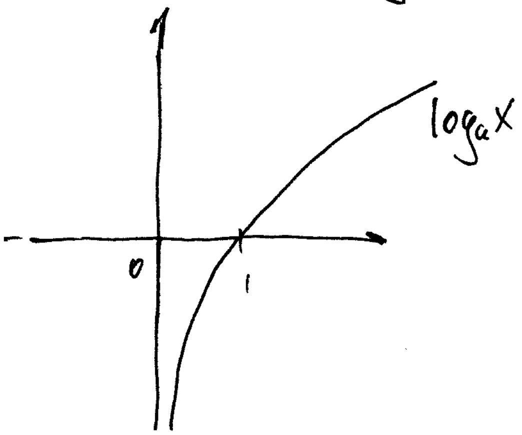


$f(x) = \left(\frac{1}{2}\right)^x$

Logarithmic functions (2-8)

are inverse function of the exponential functions $\boxed{a^{f(x)}=x}$

$f(x) = \log_a x$ a is a positive ~~function~~ constant

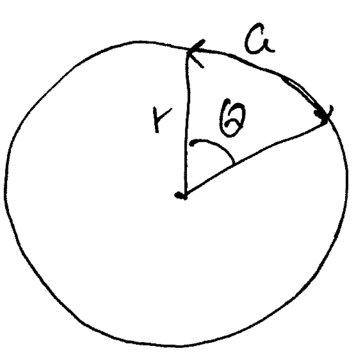


Trigonometric functions (2-9)

Angles can be measured in degrees or in radians ('rad')

$\pi \cdot \text{rad} = 180^\circ$

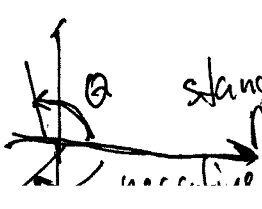
$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57{,}3^\circ$ $1^\circ = \frac{\pi}{180} \text{ rad} = 0.017 \text{ rad}.$



$a = r \cdot \theta$

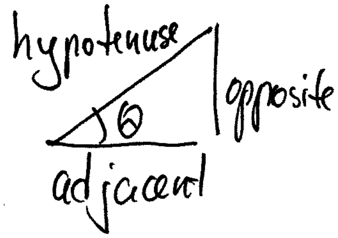
θ in radians

Degrees	0°	30°	45°	60°	90°	120°	135°	180°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π



standard position positive angles

negative angles.



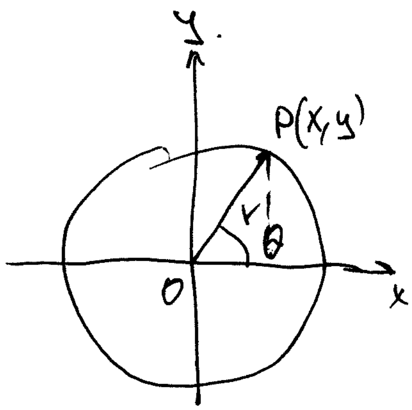
Right Triangle

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{opp}{adj}$$

$$\csc\theta = \frac{hyp}{opp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\cot = \frac{adj}{opp}.$$



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

Trigonometric identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

$$\tan^2\theta + 1 = \sec^2\theta.$$

$\sin(-\theta) = -\sin\theta$ odd

$\cos(-\theta) = \cos\theta$ even

$\sin(\theta + 2\pi) = \sin\theta$

$\cos(\theta + 2\pi) = \cos\theta$

periodic.

Addition formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

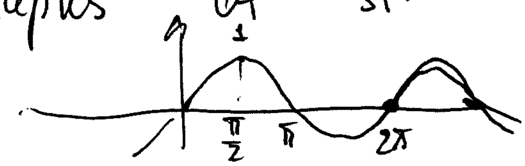
Double angle formulas

$$\sin 2x = 2\sin x \cdot \cos x$$

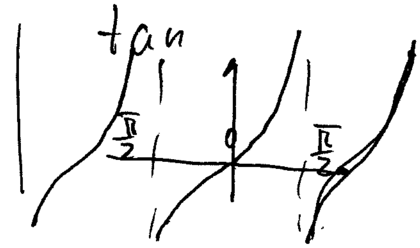
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}.$$

Graphs of sin and cos



tan



Transformation of functions

1. Translations of functions

The graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted by $c \uparrow$

The graph of $y = f(x+c)$ –||–

–||– shifted by $\overset{c}{\leftarrow}$

2 Stretching and Reflecting.

The graph of $y = c \cdot f(x)$ ~~stretch~~ shrink by c $\updownarrow$

$c > 0$

$y = f(c \cdot x)$ ~~shrink~~ stretch by c $\longleftrightarrow$

$y = -f(x)$ reflects about the x-axis

$y = f(-x)$ reflects about the y-axis.

Examples ① Transforming the root function

$y = \sqrt{x}$ $\qquad y = -\sqrt{x}$

$y = \sqrt{x} - 2$ $\qquad y = 2\sqrt{x}$

$y = \sqrt{x-2}$ $\qquad y = \sqrt{-x}$

Combinations of functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg and $\frac{f}{g}$

Sum : $(f+g)(x) := f(x) + g(x)$ $\quad \forall x \in A \cap B$.

A = Domain of f

B = Domain of g

Difference: $(f-g)(x) := f(x) - g(x)$

Product $(f \cdot g)(x) := f(x) \cdot g(x)$ and $\frac{f}{g}(x) := \frac{f(x)}{g(x)}$

The domain of $f \cdot g$ is $A \cap B$ (3-3)

The domain of $\frac{f}{g}$ is $\{x \in A \cap B \mid g(x) \neq 0\}$.

Composition of functions

Given two functions f and g, the composite function $f \circ g$ is defined by

$$(f \circ g)(x) := f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g s.t. $g(x)$ is in the domain of f.

Examples $f(x) = x^2$ and $g(x) = x + 2$

Decomposition of function (3-4)

is the process where one find the presentation of a function f as a composition of simpler functions $f = f_1 \circ f_2 \circ \dots \circ f_n$, e.g.

$$f(x) = \cos^2(x+3)$$

$$f = f_1 \circ f_2 \circ f_3$$

$$f_3(x) = x + 3$$
$$f_2(x) = \cos x$$
$$f_1(x) = x^2$$

More about exponential functions. (3-5)

$f(x) = a^x$ what if x is irrational?

$x > 0$ $\quad \frac{n}{m} < x < \frac{n+1}{m}$ $\quad a^{\frac{n}{m}} = \sqrt[m]{a^n} < f(x) < \sqrt[m]{a^{n+1}}$

It can be shown that there exactly one $f(x)$.

<u>Laws of exponent:</u>

If a and b are positive numbers and x and y are any real numbers, then

$$a^{x+y} = a^x a^y, \quad a^{x-y} = \frac{a^x}{a^y}, \quad (a^x)^y = a^{x \cdot y}$$

$$(ab)^x = a^x b^x.$$

<u>Exponential models</u> (3-6)

<u>Example:</u> The half-life of strontium-90 (^{90}Sr) is 25 years. (half of any given quantity of ^{90}Sr will disintegrate in 25 years)

If a sample of ^{90}Sr has a mass of 24mg, find an expression for the mass $m(t)$ that remains after t years.

$m(0) = 24$ $\quad m(50) = \frac{1}{2^2} \cdot 24 \quad \ldots$

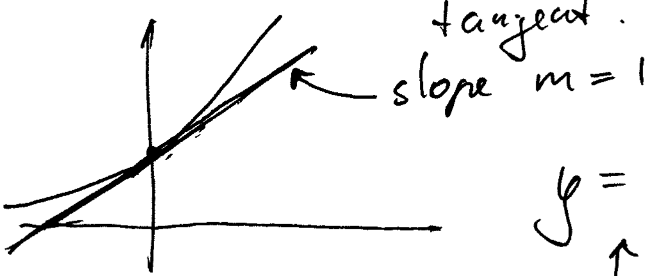
$m(25) = \frac{24}{2} = 12$

$$m(t) = \frac{24}{2^{t/25}} = 24 \cdot \left(2^{-\frac{1}{25}}\right)^t$$

exponential function with base $2^{-\frac{1}{25}}$

The number e

(3-7)



$y = e^x$ $\quad e \approx 2.718$

↑ natural exponential function

Inverse functions

A function f is called a one-to-one function if it never takes on the same value twice, that is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Horizontal line test : A function is one-to-one if and only if no horizontal line intersects its graph more than once.

(3-8)

Example. Is the function $f(x) = x^3$ one-to-one ?

Given a function $f : A \longrightarrow B$ $\left(\begin{array}{l}A \text{ is a domain}\\ B \text{ is a range of } f\end{array}\right)$

we say that a function g is an inverse of f if $g : B \longrightarrow A$ $\left(\begin{array}{l}B \text{ is a domain of } g\\ A \text{ is a range of } g\end{array}\right)$ and

$$\underset{\forall x \in A}{g(f(x)) = x}, \quad f(g(y)) = y \quad \forall y \in B.$$

f is one-to-one $\Longleftrightarrow$ the inverse of f exists

The inverse of f is denoted by f^{-1}

$$\boxed{f^{-1}(x) \neq \frac{1}{f(x)}}$$

Examples of functions and their inverses (3-9)

$f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ on $A = B = (-\infty; \infty)$

$f(x) = x + 2$ and $g(x) = x - 2$.

$f(x) = x^2$ and $g(x) = \sqrt{x}$ on $A = [0, \infty)$.
$B = [0, \infty)$

How to find the inverse of a one-to-one function

1. Write $y = f(x)$
2. Solve this equation for x in terms of y
3. To express f^{-1} as a function of x, interchange x and y.

<u>Logarithmic Functions</u> (3-10)

if $a > 0$ and $a \neq 1$ then $f(x) = a^x$ is either decreasing or increasing

so it is one-to-one.
It has an inverse function f^{-1} which is called the <u>Logarithm function</u> with base a and is denoted $\log_a$

So $\log_a x = y \iff a^y = x$

We have $\log_a(a^x) = x \quad \forall x \in \mathbb{R}$

$a^{\log_a x} = x \quad \forall x > 0$

'$\forall$' = 'for every'

Laws of Logarithms if x and y are positive

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \cdot \log_a x$

Natural logarithm if $a = e$

$$e^{\ln x} = x$$

$$\ln(e) = 1$$

Using the laws of logarithms express $\ln(a) + \frac{1}{3}\ln(b)$ as a single logarithm

$$\ln(a) + \frac{1}{3}\ln(b) = \ln(a) + \ln(b^{\frac{1}{3}}) = \ln(a \cdot \sqrt[3]{b})$$

Using the Laws of Logarithms

Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm

We have $\ln a + \frac{1}{2}\ln b = \ln a + \ln b^{\frac{1}{2}} = \ln a + \ln\sqrt{b} =$
$= \ln(a\sqrt{b})$.

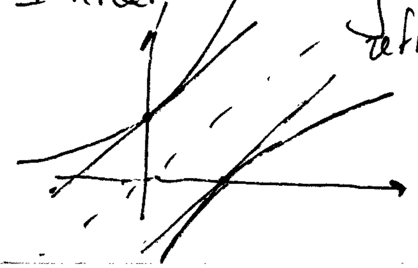
Change of base formula:

For any positive number a $(a \neq 1)$, we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Indeed, let $y = \log_a x$, then $a^y = x$ so that

reflecting about $y = x$.



$$\ln a^y = \ln x$$

$$y \cdot \ln a = \ln x \Rightarrow y = \frac{\ln x}{\ln a}$$

4-1

A tangent line to a curve is a line that touches the curve in the same direction.

Example: Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

Choose a nearby point $Q(x, x^2)$ on the parabola and compute the slope of the line PQ.

We assume $x \neq 1$ so that $Q \neq P$.

Then $m_{PQ} = \frac{x^2 - 1}{x - 1}$ is the slope of PQ. $\boxed{\frac{y_1 - y_2}{x_1 - x_2}}$

The slope of the tangent line is the slope m_{PQ} as $Q \to P$

$\lim_{Q \to P} m_{PQ} = m$ computing we obtain $m = 2$.

so the equation is $y = 2x - 1$.

The Velocity

Example Suppose that a ball is dropped from 450m above the ground. Find the velocity of the ball after 5 sec.

Fact The distance fallen by a freely falling body is proportional to the square of the time it has been falling

$$s(t) = 4.9t^2$$

in $\frac{1}{10}$th sec.

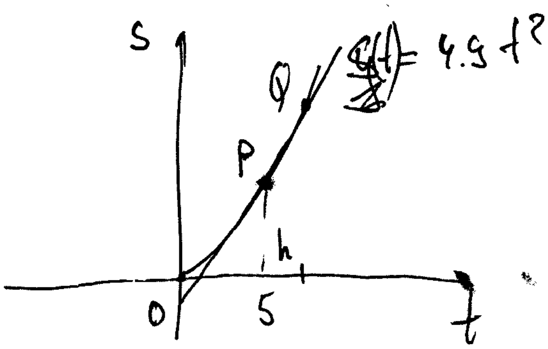
$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5.1) - s(5)}{0.1} =$$

$$= \frac{4.9\cdot(5.1)^2 - 4.9(5)^2}{0.1} =$$

$$= 49,49 \text{ m/s.}$$

The velocity at $t=5$ is the limiting value of these average velocities as 'time elapsed' $\to 0$.

$$m_{PQ} = \frac{4.9\cdot(5+h)^2 - 4.9\cdot 5^2}{(5+h) - 5} \quad \text{as } h \to 0.$$



The velocity at $t=5$ must be the same as the slope m_{PQ} of the tangen line at P.

Question If a ball is thrown into the air (4-4)
with a velocity of 40 ~~ft~~/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.
Find the average velocity for the time period beginning when $t = 2$ and lasting $\frac{1}{2}$ sec.

$t = 2$: $80 - 64 = 16$ ft. $\qquad \dfrac{16}{\frac{1}{2}} = 32$ ft/s.

$t = 2 + \frac{1}{2}$: $100 - 100 = 0$ ft.

The limit of a function (4-5)

We write $\lim_{x \to a} f(x) = L$

and say 'the limit of $f(x)$ as x approaches a, equals L'
if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (but not equal to a).

Also, $f(x) \to L$ as $x \to a$
'$f(x)$ approaches L as x approaches a'

Examples: $\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$ (4-6)

$\lim_{x \to 0} \sin \frac{\pi}{x}$ doen't exisd

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

$\lim_{t \to 0} H(t)$ doent exisd.

If we assume that $t \to 0$ from the left ($t \to 0^-$)

then $\lim_{t \to 0^-} H(t) = 0$

—11— righ ($t \to 0^+$)

then $\lim_{t \to 0^+} H(t) = 1$.

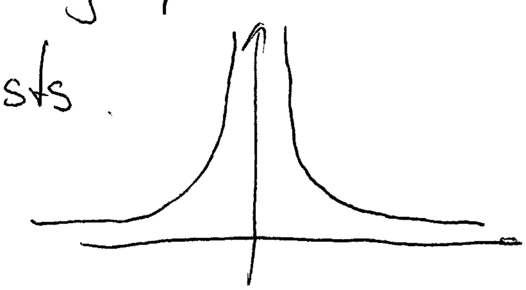
We write $\lim_{x \to a^-} f(x) = L$ (4-7)

and say the left-hand limit of $f(x)$ as x approaches a is equal to L.

One side limits from a graph (examples).

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

(doesn't exist).



Limit Laws

(4-8)

Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and let c be a constant. Then

1. $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} (c f(x)) = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} (f(x) g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.

(4-9)

6. $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$ where n is a positive integer
7. $\lim_{x \to a} c = c$ 8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (if n is even we assume $a > 0$)
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ if n is even we assume that $\lim_{x \to a} f(x) > 0$.

Example $\lim_{x\to 5}(2x^2-3x+4) =$

$$= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4$$

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

Limit Laws

(5-8)

Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and let c be a constant. Then

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Example $\lim_{x \to 5} (2x^2 - 3x + 4) =$ (5–13)

$= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4$

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Example (Direct substitution doesn't work in some cases) (5–14)

$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$ but $\frac{x^2-1}{x-1} = f(x)$ is not defined at $x = 1$.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided the limit exist.

Example Find $\lim_{x \to 1} g(x)$ where $g(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$

$g(x) = x + 1$ if $x \neq 1$ so

$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = 2.$

Example Find $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$

Let $F(h) = \frac{(3+h)^2 - 9}{h}$.

$F(h) = \frac{6h + h^2}{h} = 6 + h$ so $\lim_{h \to 0} F(h) = 6$.

Find $\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$ ... multiply both sides by $\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}-3}$

Fact: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Example $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} x = 0$.

If $f(x) \leq g(x)$ when x is near a (except a) and the limits of f and g both exist as $x \to a$ then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

(The squeeze theorem) If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$.

Example : Show that $\lim\limits_{x \to 0} x^2 \sin \frac{1}{x^2} = 0$ (5-7)

$$-1 \leq \sin \frac{1}{x^2} \leq 1.$$

$$-x^2 \leq x^2 \sin \frac{1}{x^2} \leq x^2$$

$\lim\limits_{x \to 0} x^2 = 0$ and $\lim\limits_{x \to 0} (-x^2) = 0$

~~so~~ so by the squeese theorem we are done.

For some functions the limit as $x \to a$ can be found by calculating the value of the function at $x = a$. Such functions are called continuous at a. (5-8)

A function f is called continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

(Here we assume that

- $f(a)$ is defined
- $\lim\limits_{x \to a} f(x)$ exists
- $\lim\limits_{x \to a} f(x) = f(a)$)

Assume f is defined ~~near a~~ on an (5-9)
open interval containing a (except perhaps at a).
Then f is called <u>discontinuous</u> at a if
f is not continuous at a.

Discontinuities from a graph, <u>examples</u>.

We also have:
f is continuous from the right at a if
$$\lim_{x \to a^+} f(x) = f(a)$$
–11– from the left at a if
$$\lim_{x \to a^-} f(x) = f(a)$$

---

f is continuous on an interval if (5-10)
it is continuous at every number in
the interval. If f is defined only on one
side of an endpoint of the interval, then
by continuity at the endpoint we mean
continuity from the right or continuous
from the left.

Example of a function
$$f(x) = 2 - \sqrt{4 - x^2}.$$

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a: (5-11)

1. $f+g$ 2. $f-g$ 3. $c \cdot f$

4. $f \cdot g$ 5. $\frac{f}{g}$ if $g(a) \neq 0$.

Example (a) Any polynomial function is cont. everywhere.

(b) Any rational function is cont. whenever it is defined.

The following types of functions are continuous at every number in their domains: (6-1)

polynomials
rational functions
root functions
trigonometric functions
exponential functions
logarithmic functions

<u>Fact</u>: If f is cont. at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$

<u>Fact</u>: If g is cont. at a and f is cont. at $g(a)$, then the composite $f \circ g$ is cont. at a. (6-2)

<u>Example</u> Where are the following functions continuous? $F(x) = \ln(1 + \cos x)$

$F(x)$ is continuous whenever $1 + \cos x > 0$

So F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values.

The intermediate value theorem

(6-3)

Suppose that f is cont. on $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$.

Then there exists a number C in $[a,b]$ such that $f(c) = N$.

Example: Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2

$$f(1) = -1 < 0$$
$$f(2) = 12 > 0$$

Infinite Limits

(6-4)

$$\lim_{x \to 0} \frac{1}{x^2} = \infty \qquad \text{it doen't exist.}$$

The notation $\lim_{x \to a} f(x) = \infty$ means that the limit doesn't exist and the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a)

Similarly we have $\lim_{x \to a} f(x) = -\infty$.

large $\leftrightarrow$ smaller.

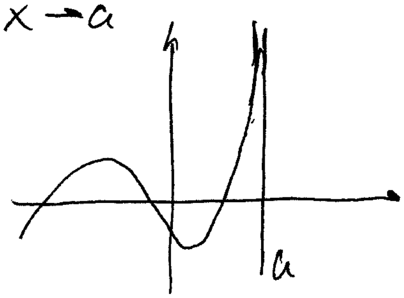
Similarly for left (righ) limits $\lim_{x \to a^+} f(x) = -\infty$

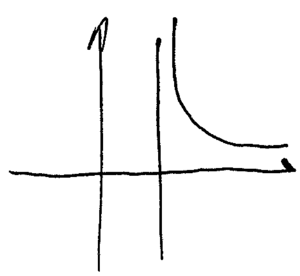
Asymptotes

The line $x = a$ is called a vertical asymptote of the graph $y = f(x)$ if at least one of the following is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$





$x \to a^-$ ∞ $\qquad$ $x \to a^+$ $\neq \infty$

6-6

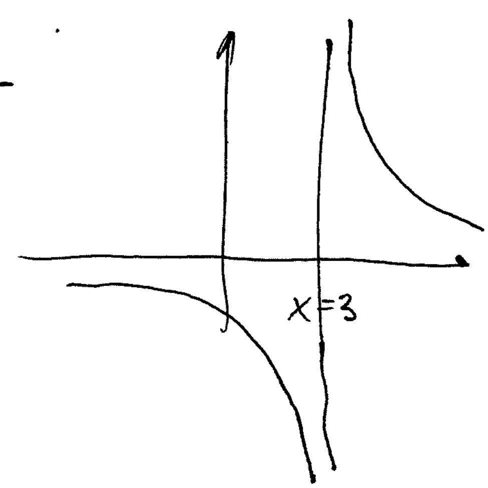
Example: Find $\lim_{x \to 3^+} \frac{2x}{x-3}$ $\qquad$ $\lim_{x \to 3^-} \frac{2x}{x-3}$

as $x \to 3^+$ $\quad x - 3 \to 0^+$ and $2x \to 6$.

so $\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$

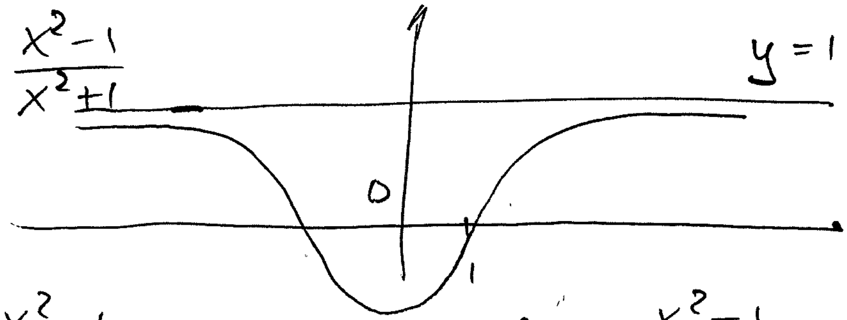
as $x \to 3^-$ $\quad x - 3 \to 0^-$

so $\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$



Limits at infinity (6-7)

$$f(x) = \frac{x^2-1}{x^2+1}$$



So, $\displaystyle\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$ $\qquad$ $\displaystyle\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1.$

Let f be a function defined on some interval (a, ∞). Then

$\displaystyle\lim_{x\to\infty} f(x) = L$ means that the

values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either (6-8)

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

All limits laws except for 9 and 10 are also valid if '$x \to a$' is replaced by '$x \to \infty$' or '$x \to -\infty$'

For instance, $\displaystyle\lim_{x\to\infty} \frac{1}{x^n} = 0$ and $\displaystyle\lim_{x\to-\infty} \frac{1}{x^n} = 0.$

Example $\qquad \displaystyle\lim_{x\to-\infty} e^x = 0.$

Finally, $\displaystyle\lim_{x\to\infty} f(x) = \infty$ means that $f(x)$ becomes large as x becomes large

Example $\lim_{x\to\infty}(x^2-x) \neq \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x$ (6-9)

wrong $\infty - \infty$ as ∞ is not a number.

Find $\lim_{x\to\infty} \frac{x^2+x}{3-x} = \lim_{x\to\infty} \frac{x+1}{\frac{3}{x}-1} = -\infty$.

Recall that the ~~tang~~ slope m of tangent line at $P(a, f(a))$ to the graph $y=f(x)$ is given by

$$m = \lim_{Q\to P} m_{PQ} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$Q(x, f(x))$ provided that this limit exists.

If $h = x-a$ then $x = a+h$ and (6-10)

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

so $$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Recall also that if an object moves along a straight line according $s = f(t)$, then the velocity at the moment a is

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

The derivative of a function f at a number a, denoted by $f'(a)$, is (6-11)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

or equivalently $(x = a+h)$

$$f'(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a}.$$

Examples of computations for polynomials.

The tangent line to $y = f(x)$ at $x = a$ has the slope $f'(a)$. (6-12)

We also write ~~Δy~~ $\Delta y = f(x_2) - f(x_1)$ and $\Delta x = x_2 - x_1$ so that

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{\Delta y}{\Delta x}$$

The Derivative as a function.

We assume now that 'a' changes

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ is a new function

it assigns to x the value $f'(x)$.

Example If $f(x) = x^2$, find the derivative of f. State the domain of f' (6-13)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

A function f is differentiable at a (7-1)
if $f'(a)$ exists.
It is differentiable on an open interval (a,b) or (a,∞), $(-\infty,a)$ if it is differentiable at every number in the interval.

<u>Example</u> Where is the function $f(x)=|x|$ differentiable?

<u>Solution</u>: If $a>0$, then

$$f'(a)=\lim_{h\to 0}\frac{|a+h|-|a|}{h}=\lim_{h\to 0}\frac{(a+h)-a}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

If $a<0$, then $f'(a)=-1$.

---

if $a=0$, then (7-2)

$$\lim_{h\to 0^+}\frac{|0+h|-|0|}{h}=1 \quad\text{and}\quad \lim_{h\to 0^-}\frac{|0+h|-|0|}{h}=-1$$

so the limit <u>doesn't</u> exists as $h\to 0$.
So $f'(0)$ doesn't exist.
Thus f is differentiable at all x except 0.

$$f'(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0.\end{cases}$$

<u>Fact</u>: If f is differentiable at a,
then f is continuous at a.
(the converse is false)

---

f is <u>not</u> differentiable at $x=a$ if

- the graph of f has a corner at $x=a$
  (it doesn't have a tangent line at $x=a$
  or has multiple tangent lines).
- the graph has a vertical tangent line at $x=a$,
  that is f is continuous at $x=a$ and
  $$\lim_{x\to a} |f'(x)| = \infty .$$
- f is not continuous at $x=a$.

<u>Higher derivatives</u>

$(f')' = f'' \ldots$

Another notation $\quad f'(x) = \dfrac{df(x)}{dx} \qquad f''(x) = \dfrac{d^2 f(x)}{dx^2}$

$$f^{\overbrace{\prime\cdots\prime}^{n}}(x) = \frac{d^n f(x)}{dx^n} \qquad f^{\overbrace{\prime\cdots\prime}^{n}}(x) = f^{(n)}(x).$$

We know that $\quad v(t) = \dfrac{ds}{dt}$ velocity

$$a(t) = \frac{d^2 s}{dt^2} \quad \text{acceleration}$$

<u>Example</u>: Find $f''(x)$ of $f(x) = x^3 - x$

$f'(x) = 3x^2 - 1 . \quad f''(x) = 6x .$

(7-5)

Recall that $f'(x)$ represents the slope of the curve $y=f(x)$ at the point $(x, f(x))$
So the derivative of f tells where a function is increasing or decreasing.

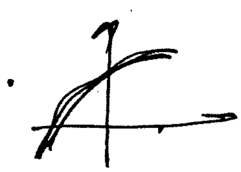
If $f'(x) > 0$ on an interval, then f is increasing on that interval.

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

(7-6)

If $f''(x) > 0$ then f' is an increasing function
So the slopes of the tangent lines of $y=f(x)$ increase from left to right
So our curve $y=f(x)$ bends upward
Such a curve is called <u>concave upward</u>

If $f''(x) < 0$, then curve $y=f(x)$ bends downward
so its graph is called <u>concave downward</u>.



The point where the curve $y=f(x)$ changes from C.U to C.D. is called an <u>inflection point</u>.

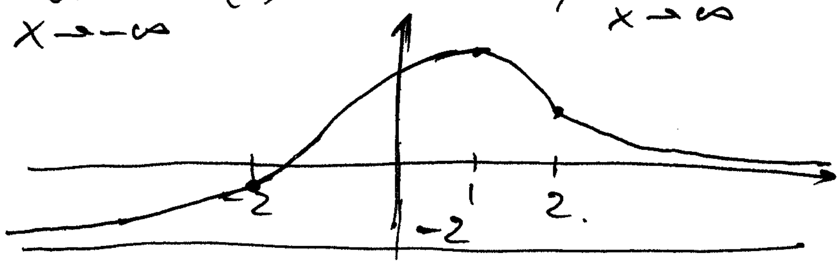
<u>Example</u> Sketch a possible graph of a function that satisfies

(i) $f'(x)>0$ on $(-\infty;1)$, $f'(x)<0$ on $(1,\infty)$

(ii) $f''(x)>0$ on $(-\infty,-2)$ and $(2,\infty)$

$f''(x)<0$ on $(-2,2)$

(iii) $\lim_{x\to-\infty} f(x)=-2$, $\lim_{x\to\infty} f(x)=0$



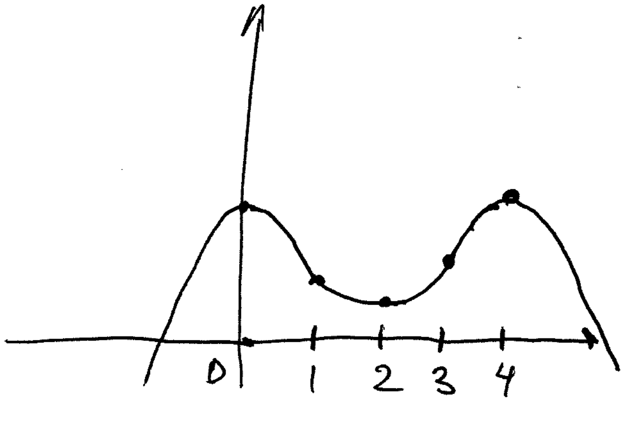
Sketch the graph

$f'(0)=f'(2)=f'(4)=0$

$f'(x)>0$ if $x<0$ or $2<x<4$

$f'(x)<0$ if $0<x<2$ or $x>4$

$f''(x)>0$ if $1<x<3$, $f''(x)<0$ if $x<1$ or $x>3$



We can use the definition of the derivative to find the derivative of a function.

Example

Find the derivative of $y = \frac{1}{\sqrt{x}}$ and use this to find the equation of the tangent line to the curve at $x = 4$.

Solution: Set $f(x) = \frac{1}{\sqrt{x}}$ and compute the difference quotient:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\ &= \frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x} \cdot \sqrt{x+h}}\end{aligned}$$

The standard thing to do to simplify an expression with a difference of radicals is to rationalize, so we try that. Multiply by $\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{1}{h} \frac{x - (x+h)}{\sqrt{x} \cdot \sqrt{x+h} \cdot (\sqrt{x} + \sqrt{x+h})} \\ &= \frac{1}{h} \frac{-h}{\sqrt{x} \cdot \sqrt{x+h} \cdot (\sqrt{x} + \sqrt{x+h})} \\ &= \frac{-1}{\sqrt{x} \cdot \sqrt{x+h} \cdot (\sqrt{x} + \sqrt{x+h})}\end{aligned}$$

and we see that this function is continuous at $h = 0$, so the limit exists:

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-1}{\sqrt{x} \cdot \sqrt{x+h} \cdot (\sqrt{x} + \sqrt{x+h})} \\ &= \frac{-1}{2x^{3/2}} \\ &= -\frac{1}{2} x^{-3/2}\end{aligned}$$

Now, at $x = 4$ the derivative is

$$f'(4) = -\frac{1}{2}(4)^{-3/2} = -\frac{1}{2}(\frac{1}{8}) = -\frac{1}{16}$$

which is the slope of the tangent line to $y = \frac{1}{\sqrt{x}}$ at $(4, \frac{1}{2})$. Thus the equation of the tangent line is

$$y = \frac{-1}{16}x + b$$

where b is obtained by plugging in the point $(4, \frac{1}{2})$: $b = \frac{1}{2} + \frac{1}{16}4 = \frac{3}{4}$; giving final answer

$$y = -\frac{1}{16}x + \frac{3}{4}.$$

We have seen that

$$\frac{d}{dx}x^2 = 2x$$

and

$$\frac{d}{dx}x^{-1/2} = \frac{-1}{2}x^{-3/2}$$

In fact, the following *power rule* is true.

$$\boxed{\text{Power rule: for any } n \in \mathbb{R},\ \frac{d}{dx}x^n = nx^{n-1}.}$$

Example

$$\frac{d}{dx}x^7 = 7x^6,$$

$$\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}, \text{ and}$$

$$\frac{d}{dx}x^{\pi} = \pi x^{\pi-1}.$$

It works even when the power is in disguise:

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

. *Let f and g be differentiable functions on a common domain . Then* 8-3

- *for any constant c,*

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

- *and*

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

It follows that $(f-g)' = f' - g'$ as well.

The reason this happens is that it's true for the difference quotient:

$$\frac{cf(x+h)-cf(x)}{h} = c\frac{f(x+h)-f(x)}{h}$$

and

$$\frac{(f+g)(x+h)-(f+g)(x)}{h} = \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \frac{\Delta f}{\Delta x} + \frac{\Delta g}{\Delta x}.$$

Since it's true of every difference quotient, it is also true of the limit.

8-4

Example

$$\begin{aligned}\frac{d}{dx}(4x^7-2x^3-1) &= \frac{d}{dx}(4x^7) - \frac{d}{dx}(2x^3) - \frac{d}{dx}(1)\\ &= 4\frac{d}{dx}x^7 - 2\frac{d}{dx}x^3 - 0\\ &= 28x^6 - 6x^2\end{aligned}$$

Derivative of e^x 8-5

Suppose $a > 0$ and consider the function $f(x) = a^x$. Its derivative has an interesting property.

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h} = a^x\left(\frac{a^h-1}{h}\right)$$

Taking the limit as $h \to 0$, we have

$$f'(x) = a^x \lim_{h\to 0}\frac{a^h-1}{h} = a^x f'(0)$$

That is to say, the rate of change of this function is proportional to the value of the function.

This is a really odd property; let's interpret it from an example.

Example . Suppose we have a population of bacteria, which doubles in population every hour. If our initial population is 1g, then after t hours the population is

$$p(t) = 2^t$$

grams. What is the rate of change? The population doubles every hour; so the rte of change depends on the size of the population. If you have a population of 2^t at time t, then at $t+1$ the population has doubled to 2^{t+1}, giving an average rate of change of $2^{t+1} - 2^t = 2^t(2-1) = 2^t$ (divided by 1).

8-6

But this is NOT the instantaneous rate of change (since the graph is not a straight line from $(t, 2^t)$ to $(t+1, 2^{t+1})$); we've deduced above that the instantaneous rate of change is

$$p'(t) = 2^t p'(0)$$

whatever $p'(0)$ is (a number).

Graphically: it says the slope of the function a^x is proportional to its height at every point.

By sketching the graphs of various exponential functions, we deduce that there is a unique a such that the derivative of a^x at $x = 0$ is equal to 1. This number is

$$e$$

and it is around 2.71828183.

Example 8-7

$$\begin{aligned}
\frac{d}{dx}(e^{x+3} - 5x^2) &= \frac{d}{dx}(e^{x+3}) - \frac{d}{dx}(5x^2) \\
&= \frac{d}{dx}(e^x e^3) - 5\frac{d}{dx}x^2 \\
&= e^3 \frac{d}{dx} e^x - 10x \\
&= e^3 e^x - 10x \\
&= e^{x+3} - 10x
\end{aligned}$$

Product rule 8-8

Let f and g be differentiable functions on a common domain Then the product function fg is also differentiable, and its derivative is given by

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

We work on the difference quotient and apply a brilliant trick:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} g(x) \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}$$

since upon taking the limit as $h \to 0$, we see that $f(x+h) \to f(x)$, and the difference quotients go to the derivatives. So the limit exists and fg is differentiable with the stated derivative.

Example

$$\frac{d}{dx}(x^2e^x) = x^2\left(\frac{d}{dx}e^x\right) + \left(\frac{d}{dx}x^2\right)e^x = x^2e^x + 2xe^x = e^x(x^2+2x)$$

The rule extends to products of 3 and more terms, via

$$\begin{aligned}(fgh)'(x) &= f(x)(gh)'(x) + f'(x)(gh)(x) \\ &= f(x)(g(x)h'(x) + g'(x)h(x)) + f'(x)g(x)h(x) \\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)\end{aligned}$$

Example

$$\begin{aligned}\frac{d}{dx}(x^{4/3} + x^{1/3})e^x &= \frac{d}{dx}x^{1/3}(x+1)e^x \\ &= x^{1/3}(x+1)e^x + x^{1/3}e^x + \frac{1}{3}x^{-2/3}(x+1)e^x \\ &= x^{-2/3}e^x\left(x(x+1) + x + \frac{1}{3}(x+1)\right) \\ &= e^x\frac{x^2 + \frac{7}{3}x + \frac{1}{3}}{x^{2/3}}\end{aligned}$$

which you might also simplify further

The quotient rule

8-10

Let f and g be differentiable functions on a common domain. Then the quotient function $\frac{f}{g}$ is differentiable everywhere that $g(x) \neq 0$, and its derivative is given by

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Note that unlike the product rule, which was symmetric in f and g, this rule is certainly NOT symmetric! Even the order of the terms in the numerator is vitally important.

Let's work on the difference quotient; we use the same clever trick as in the product rule.

$$\begin{aligned}\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) &= \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right) \\ &= \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right) \\ &= \frac{1}{g(x)g(x+h)}\left(g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}\right)\end{aligned}$$

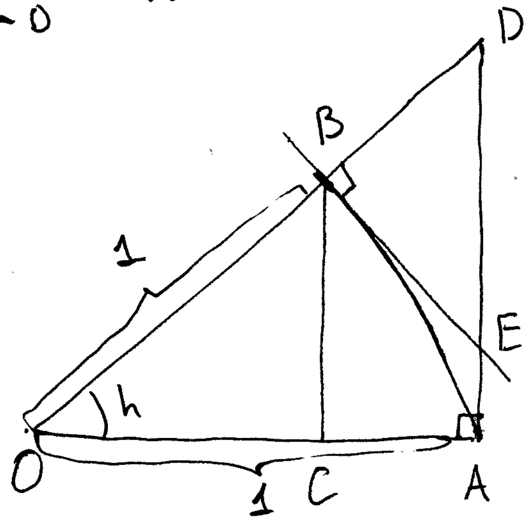
Taking the limit as $h \to 0$, we derive the familiar formula.

Example

$$\frac{d}{dx}\frac{x^2}{x+1} = \frac{(x+1)(2x) - x^2(1)}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}.$$

An important limit. (9-5)

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$



Assume that $0 < h < \frac{\pi}{2}$.

$|BC| = |OB| \cdot \sin(h) = \sin(h)$

$|BC| < |AB| < \text{arc } AB = h$

$\sin(h) < h$, so $\frac{\sin(h)}{h} < 1$.

$h = \text{arc } AB < |AE| + |EB| <$

$< |AE| + |ED| =$

$= |AD| = |OA| \cdot \tan(h)$

$= \tan(h) = \frac{\sin(h)}{\cos(h)}$.

So $\cos(h) < \frac{\sin(h)}{h}$.

Apply the Squeeze Theorem: $\lim_{h\to 0^+} \cos(h) = 1$ and $\lim_{h\to 0^+} 1 = 1$

So $\lim_{h\to 0^+} \frac{\sin(h)}{h} = 1$.

Derivative of Trigonometric Functions 9-2.

Let $f(x) = \sin x$. How to find $f'(x)$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} =$$

$$= \lim_{h\to 0} \left[\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \right] =$$

$$= \lim_{h\to 0} \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} + \lim_{h\to 0} \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}.$$

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = \lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1}\right) =$$ (9-3)

$$= \lim_{h\to 0} \frac{\cos^2(h)-1}{h\cdot(\cos(h)+1)} = \lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)} =$$

$$= -\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\frac{\sin(h)}{\cos(h)+1} =$$

$$= -\lim_{h\to 0}\frac{\sin(h)}{h}\cdot\lim_{h\to 0}\frac{\sin(h)}{\cos(h)+1} = 0.$$

So $f'(x) = \cos(x)$.

<u>Example</u> Compute $(x^2\sin(x))'$ (9-4)

$$(x^2\sin(x))' = (x^2)'\cdot\sin(x) + x^2\cdot(\sin(x))' =$$

$$= 2x\cdot\sin(x) + x^2\cdot\cos(x).$$

$$\boxed{(\cos(x))' = -\sin(x)} \qquad (\tan(x))' = \frac{1}{\cos^2 x} = \sec^2 x$$

Finally,

$$(\csc(x))' = -\csc(x)\cdot\cot(x)$$

$$(\sec(x))' = \sec(x)\cdot\tan(x)$$

$$(\cot(x))' = -\csc^2 x.$$

Example Find $f(x)'$ where $f(x) = \dfrac{\sec(x)}{1+\tan(x)}$ 9-5

$$f'(x) = \frac{(1+\tan(x))\cdot \sec(x)' - \sec(x)\cdot(1+\tan(x))'}{(1+\tan(x))^2} =$$

$$= \frac{(1+\tan(x))\sec(x)\tan(x) - \sec(x)\cdot\sec^2(x)}{(1+\tan(x))^2} =$$

$$= \frac{\sec(x)\left(\tan(x) + \tan^2(x) - \sec^2(x)\right)}{(1+\tan(x))^2} =$$

$$= \frac{\sec(x)\cdot(\tan(x)-1)}{(1+\tan(x))^2}.$$

9-6

$f(x) = \cos(x)$ $\quad f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$ $\quad f^{(3)}(x) = \sin(x)$

$f^{(4)}(x) = \cos(x)$ $\quad \ldots$

So
$$f^{(n)}(x) = \begin{cases} \cos(x) & \text{if } n \text{ is a multiple of } 4 \\ -\sin(x) & \text{if } n-1 \text{ ―\texttt{"}―} \\ -\cos(x) & \text{if } n-2 \text{ ―\texttt{"}―} \\ \sin(x) & \text{if } n-3 \text{ ―\texttt{"}―} \end{cases}$$

The Chain Rule (9-7)

If g is differentiable at x and
f is differentiable at $g(x)$, then
the composite function $F = f \circ g$ defined
by $F(x) = f(g(x))$ is differentiable at x
and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x).$$

if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

<u>Example</u>: Find $(\sin(x^2))'$ (9-8)

$\sin(x^2) = f(g(x))$ where $f(u) = \sin(u)$
$u = g(x) = x^2$

So $\sin(x^2)' = f'(g(x)) \cdot g'(x) = \cos(x^2) \cdot 2x.$

<u>Example</u>: Find $((x^3-1)^{100})'$

$(x^3-1)^{100} = f(g(x))$ where $f(u) = u^{100}$
$u = g(x) = x^3-1$

$((x^3-1)^{100})' = 100 \cdot u^{99} \cdot (x^3-1)' = 100 \cdot (x^3-1)^{99} \cdot (3x^2) =$
$= 300 \cdot x^2 \cdot (x^3-1)^{99}$

$(a^x)' = ((e^{\ln a})^x)' = (e^{(\ln a) \cdot x})' =$
$= \ln a \cdot e^{(\ln a) \cdot x} = \ln a \cdot a^x$

Parametric Curves

Consider a function $y = h(x)$

Suppose that x and y are given as functions of a variable t (called a parameter)

$$x = f(t) \quad \text{and} \quad y = g(t)$$

(called parametric equations)

The graph is called a parametric curve.

By the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{So} \quad \boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \quad \text{if } \frac{dx}{dt} \neq 0$$

This allows us to find the slope $\frac{dy}{dx}$

of the tangent to a parametric curve without having to eliminate the parameter t.

The curve has a horizontal tangent when $\frac{dy}{dt} = 0$ (if $\frac{dx}{dt} \neq 0$)

it has a vertical tangent when $\frac{dx}{dt} = 0$ (if $\frac{dy}{dt} \neq 0$)

(9-11)

Example Find an equation of the tangent line to the parametric curve

$$x = 2\sin 2t \quad , \quad y = 2\sin t \quad , \quad 0 \le t \le 2\pi$$

at the point $(\sqrt{3}, 1)$.

Where does this curve have horizontal or vertical tangents?

$$\frac{dy}{dx} = \frac{\frac{d}{dt}(2\sin(t))}{\frac{d}{dt}(2\sin 2t)} = \frac{2\cos(t)}{2\cos(2t)\cdot 2} = \frac{\cos t}{2\cos 2t}$$

$(\sqrt{3}, 1)$ corresponds to $t = \pi/6$ so $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{\cos(\frac{\pi}{6})}{2\cos(\frac{\pi}{3})} = \frac{\sqrt{3}/2}{2\cdot\frac{1}{2}} = \frac{\sqrt{3}}{2}$.

An equation is $y - 1 = \frac{\sqrt{3}}{2}(x - \sqrt{3})$ or

$y = \frac{\sqrt{3}}{2}x - \frac{1}{2}$.

(9-12)

$\frac{dy}{d\cancel{t}} = 0$ if $\cos t = 0$ (and $\cos 2t \neq 0$)

so when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\Rightarrow$ the curve has horizontal tangents at $(0, 2)$ and $(0, -2)$.

$\frac{dx}{dt} = 0 = 4\cos 2t$ (and $\cos t \neq 0$)

so when $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

$\Rightarrow$ the curve has vertical tangents at $(\pm 2, \pm\sqrt{2})$.

Implicit Differentiation

(10-1)

Some functions are defined implicitly by a relation between x and y such as

$$x^2+y^2=4$$

or $$x^3+y^3=6xy$$

The first we can solve $y=\pm\sqrt{4-x^2}$ and find the derivative using the rules of diff

What about the second one?

We use the method of <u>implicit differentiation</u>: we differentiate the relation and solve it in y'.

<u>Example</u> Find y' if $x^3+y^3=6xy$ (10-2)

$$\frac{d(x^3+y^3)}{dx} = \frac{d(6xy)}{dx}$$

$$3x^2+3y^2\cdot y' = 6xy'+6y$$

$$x^2+y^2\cdot y' = 2xy'+2y$$

$$y^2\cdot y'-2xy' = 2y-x^2$$

$$(y^2-2x)\cdot y' = 2y-x^2$$

$$y' = \frac{2y-x^2}{y^2-2x}$$

Example Find y' if $\sin(x+y) = y^2 \cos x$ (10-3)

$$\cos(x+y)\cdot(1+y') = y^2(-\sin x) + (\cos x)\cdot(2yy'$$

$$\cos(x+y) + y^2 \sin x = (2y\cos x)y' - \cos(x+y)\cdot y'$$

So $$y' = \frac{y^2\sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Example Find y'' if $x^4 + y^4 = 16$

$$4x^3 + 4y^3y' = 0 \qquad y' = -\frac{x^3}{y^3}$$

$$y'' = \left(-\frac{x^3}{y^3}\right)' = -\frac{y^3\cdot 3x^2 - x^3\cdot(3y^2\cdot y')}{y^6} =$$

$$= -\frac{y^3\cdot 3x^2 - x^3\cdot 3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} = -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4+x^4)}{y^7}$$

$$= -\frac{3x^2\cdot 16}{y^7} = -48\frac{x^2}{y^7}$$

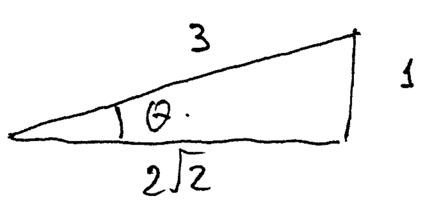
Inverse trigonometric functions and Their derivatives (10-4)

$$f(x) = \sin(x) \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

has the inverse

$$f^{-1}(x) = \sin^{-1}(x) = \arcsin(x) \qquad \text{arcsine function.}$$

- $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$
- $\tan\left(\arcsin\frac{1}{3}\right) = \tan\theta = \frac{1}{2\sqrt{2}}$ where $\sin\theta = \frac{1}{3}$



$\arcsin$ has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Let $y = \arcsin(x)$ that is $\sin y = x$

$$(\sin y)' = 1$$

$$\cos y \cdot y' = 1 \quad \text{so} \quad y' = \frac{1}{\cos y}$$

$\cos y \geq 0$ since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$\boxed{y' = \frac{1}{\sqrt{1-x^2}} \quad \text{on} \quad -1 < x < 1}$$

Example $f(x) = \arcsin(x^2 - 1)$

(a) Find the domain of f

(b) Find $f'(x)$ and the domain of $f'(x)$

(a): Since the domain of $\arcsin$ is $[-1, 1]$ the domain of f is $\{x \mid -1 \leq x^2 - 1 \leq 1\} =$

$= \{x \mid 0 \leq x^2 \leq 2\} =$

$= [-\sqrt{2}, \sqrt{2}]$

(b): Chain rule:

$$f'(x) = \arcsin(x^2-1)' = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot (2x) =$$

$$= \frac{2x}{\sqrt{2x^2 - x^4}}$$

$f(x) = \cos(x) \qquad 0 \le x \le \pi$ (10-8)

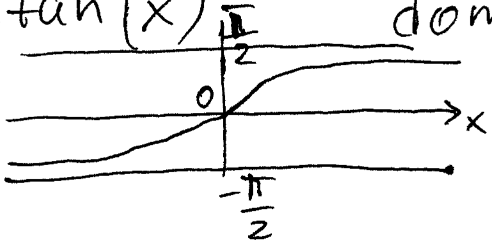
The inverse $\cos^{-1}(x)$ is called arccosine

$\underline{\arccos(x)}$ domain $[-1, 1]$, range $[0, \pi]$

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1$$

$f(x) = \tan(x) \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\arctan(x)$ domain $(-\infty; +\infty)$ range $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$$\arctan'(x) = \frac{1}{1+x^2}$$

Find $\left(x \arctan\sqrt{x}\right)'$ (10-8)

$$= x \cdot \frac{1}{1+\sqrt{x}^2} \cdot \frac{1}{2} x^{-\frac{1}{2}} + \arctan\sqrt{x} =$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}.$$

Find $\arccos\left(e^{2x}\right)' = \frac{1}{\sqrt{1-e^{4x}}} \cdot 2 \cdot e^{2x}$

Derivatives of logarithmic functions

11-1

Consider $y = \log_a x$ then $a^y = x$

Differentiating implicitly we obtain

$$\underbrace{(a^y)'}_{a^y(\ln a)\cdot y'} = (x)' = 1$$

so $y' = \dfrac{1}{a^y \ln(a)} = \dfrac{1}{x \ln(a)}$

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$$

If $a = e$, then $\boxed{(\ln x)' = \dfrac{1}{x}}$

11-2

Example . Find y' where $y = \ln(x^3+1)$

Use Chain Rule:

$$\ln(x^3+1)' = \frac{1}{x^3+1} \cdot (x^3+1)' = \frac{3x^2}{x^3+1}.$$

- Find $\dfrac{d \ln(\sin(x))}{dx}$

$$\frac{d \ln(\sin(x))}{dx} = \frac{1}{\sin x} \cdot \frac{d(\sin(x))}{dx} = \frac{1}{\sin x} \cos x = \cot x$$

- Find $(\sqrt{\ln(x)})' = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot (\ln(x))' =$

$$= \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

• Find $(\ln|x|)'$ $\qquad f(x) = \ln|x|$

$$f(x) = \begin{cases} \ln(x) & \text{if } x>0 \\ \ln(-x) & \text{if } x<0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ \frac{1}{-x}\cdot(-1) & \text{if } x<0 \end{cases}$$

So $f'(x) = \frac{1}{x}$ for all $x \neq 0$

Logarithmic Differentiation

Example let $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

Find y'.

Solution: Take logarithms of both sides

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly gives

$$\frac{1}{y}\cdot y' = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\cdot\frac{1}{3x+2}\cdot 3$$

So $y' = y\left(\dfrac{3}{4x} + \dfrac{x}{x^2+1} - \dfrac{15}{3x+2}\right) = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\dfrac{3}{4x} + \dfrac{x}{x^2+1} - \dfrac{15}{3x+2}\right)$

Steps in Logarithmic Differentiation: (11-5)

1. Take natural logarithms of both sides
2. Differentiate implicitly
3. Solve the resulting equation for y'.

Example Find $(x^{\sqrt{x}})'$. $y = x^{\sqrt{x}}$

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + \frac{1}{2\sqrt{x}} \cdot \ln x \qquad y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) =$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln(x)}{\sqrt{x}}\right)$$

The number e as a limit. (11-6)

We know that if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

so $f'(1) = 1$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} =$$

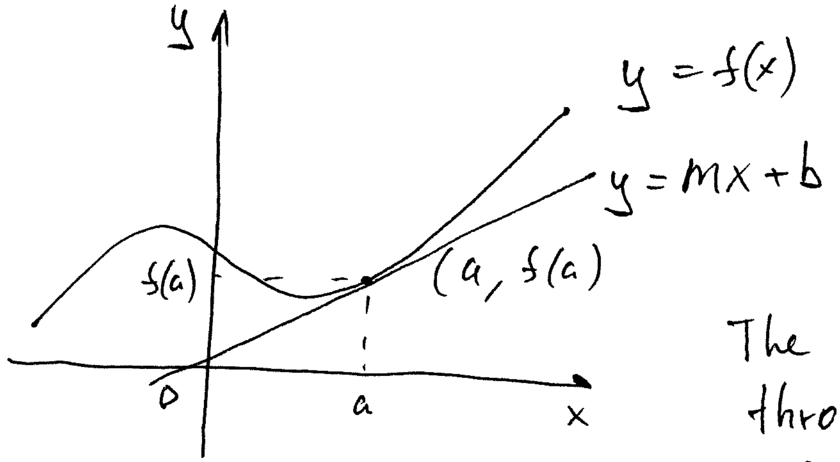
$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) =$$

$$= \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} = 1 \qquad \text{Taking } e:$$

$$e^{\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e^1 = e.$$

or $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

Linear approximations and differentials



$m = f'(a)$

The line $y = mx + b$ passes through $(a, f(a))$

Therefore $f(a) = m \cdot a + b$

So $b = f(a) - m \cdot a = f(a) - f'(a) \cdot a$

Finally, we obtain.

$$y = f'(a) \cdot x + f(a) - f'(a) \cdot a = f'(a) \cdot (x-a) + f(a)$$

is called the linear approximation of f at a.

The linear function whose graph is this tangent is called the linearization of f at a

$$L(x) = f(a) + f'(a)(x-a).$$

Example Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a=1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$

$$f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+3}}$$ so $f(1) = 2$ and $f'(1) = \frac{1}{4}$

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}$$

So $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ and $\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$

$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

Differentials

(11-9)

Let $y = f(x)$ be a differentiable function.
We introduce an independent variable dx
then $dy = f'(x)\,dx$ $\quad \left(\frac{dy}{dx} = f'(x)\right)$
dy, dx are called differentials

Meaning $\quad dx \approx \Delta x$ ~~=~~ difference of argument

$dy \approx \Delta y = f(x+\Delta x) - f(x)$ difference of values.

as it becomes smaller.

(11-10)

Example The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm.
What is the maximum error in using this value of the radius to compute the volume of the sphere?

---

r is the radius, then $V = \frac{4}{3}\pi r^3$
Let dr be the error. Then $dV = \left(\frac{4}{3}\pi r^3\right)' \cdot dr =$

$$= 4\pi r^2 \cdot dr$$

if $r = 21$ and $dr = 0.05$
we get $dV = 4\pi (21)^2 \cdot 0.05 = 277 \text{ cm}^3$.

Related Rates

(12-1)

Inflating a balloon Air is being pumped into a spherical balloon so that its volume increases at a rate $100\ cm^3/s$.
How fast is the radius of the balloon increasing when the diameter is 50 cm?

Given The rate of increase of the volume = 100

Unknown -"- of the radius when the diameter is 50 cm

V = volume   r = radius

Rates of change = derivatives

(12-2)

R.of.Change of the volume $\frac{dV}{dt} = 100\ cm^3/s$

R.of.C. of the radius $\frac{dr}{dt}$ when $r = 25cm$.

$V = \frac{4}{3}\pi r^3$ formula for the volume.

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi 25^2} \cdot 100 = \frac{1}{25\pi} \approx 0.0127\ cm/s$.

Maximum and Minimum Values

(12-3)

Let c be a number in the domain D of a funct. f

Then $f(c)$ is the

- absolute maximum value of f on D if
  $f(c) \geq f(x)$ for all x in D
- absolute minimum value of f on D if
  $f(c) \leq f(x)$ for all x in D

Absolute maximum/minimum = Global maximum/minimum

The maximum and minimum values of f are called extreme values of f.

(12-4)

The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$
  when x is near c
- local minimum value of f if $f(c) \leq f(x)$
  when x is near c

near = on some open interval containing c.

<u>Examples of graphs</u>. $f(x) = \sin x$ $\quad f(x) = x^2$

$f(x) = x^3$ (no max/min)

The Extreme Value Theorem. 12-5

. If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=$

Note that the converse is false in general: when $f'(c)=0$, f doesn't necessarily have a maximum/min at c.

Fermat's Theorem does suggest that we should start looking for exteme values of f at the numbers c where $f'(c)=0$ or where $f'(c)$ does not exist. 12-6

A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ doesn't exist.

Example Find the critical numbers of

$f(x) = x^{3/5}(4-x)$

$$f'(x) = x^{3/5}(-1) + \frac{3}{5}x^{-\frac{2}{5}}(4-x) = -x^{3/5} + \frac{3(4-x)}{5x^{2/5}} =$$

$\left(\frac{3}{2}, 0\right)$ $$= \frac{12-8x}{5x^{2/5}}$$ So $f'(x)=0$ if $12-8x=0 \Rightarrow x=\frac{3}{2}$.

If f has a local maximum or minimum at c, then c is a critical number of f (12-7)

The closed interval method to find an absolute max/min on a closed interval of a continuous f

1. Find the values of f at the critical numbers of f in (a,b)
2. Find the values $f(a)$, $f(b)$
3. The largest of the values of 1. and 2. is the absolute maximum value
   The smallest —"— absolute minimum.

Examples (12-8)

- Find critical values

$$f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$$

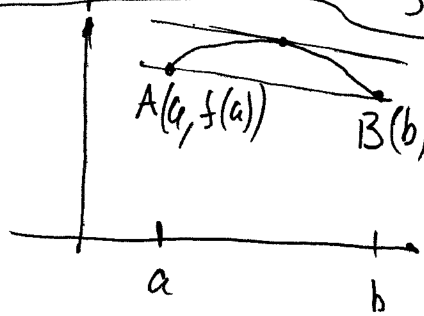
- Find the absolute maximum and minimum of $f(x) = 12 + 4x - x^2$ on $[0,5]$.

Derivatives and the shapes of curves (13-1)

The mean value theorem. If f is a differentiab[le] function on the interval $[a,b]$, then there exists a number c between a and b such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently, $f(b)-f(a) = f'(c)(b-a)$



slope $m_{AB} = \dfrac{f(b)-f(a)}{b-a}$.

Example (13-2) If an object moves in a straight line with position $s = f(t)$, then the average velocity between $t=a$ and $t=b$ is

$$\frac{f(b)-f(a)}{b-a}$$ and the velocity at $t=c$ is $f'(c)$.

So the Mean value theorem tells that at some time $t=c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity.

If a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once

The M.V. Thm can be used to show the increasing/decreasing test. (13-3)

<u>Recall</u>: If $f'(x)>0$ on an interval, then f is increasing

<u>Indeed</u>, take x_1, x_2 from the interval such that $x_1 < x_2$

Then by M.V. Thm we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now $f'(c) > 0$ and $x_2 - x_1 > 0$. So $f(x_2) - f(x_1) > 0$

$f(x_2) > f(x_1)$

This also gives us the following test to determine whether or not f has a local maximum/minimum at a critical number (13-4)

<u>The first derivative test</u> Let c be a critical number

(a) If f' changes from positive to negative at c, then f has a local maximum at c

(b) If f' changes from negative to positive at c, then f has a local minimum at c

(c) If f' does not change sign at c, then f has no local maximum or minimum at c.

Concavity

(13-5)

Recall: If f' is increasing, then f is concave upward
If f' is decreasing, then f is concave downwarc

Or (a) If $f''(x) > 0$ then f is c. u.
(b) If $f''(x) < 0$ then f is c. d.

Inflection point is the point where $f''(x) = 0$.

The second derivative test

(13-6)

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Let $f(x) = x^4 - 4x^3$ (13-7)

Find points of inflection, local max/min. and characterize concativity

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Find critical numbers

$f'(x) = 0$ so $x = 0, 3$

Evaluate f'' at critical numbers

$f''(0) = 0$ $f''(3) = 36 > 0$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimun

---

Since $f''(0) = 0$ no info by the second der. test (13-8)

But by the first der. test

Since $f'(x) < 0$ for $x < 0$ and $0 < x < 3$

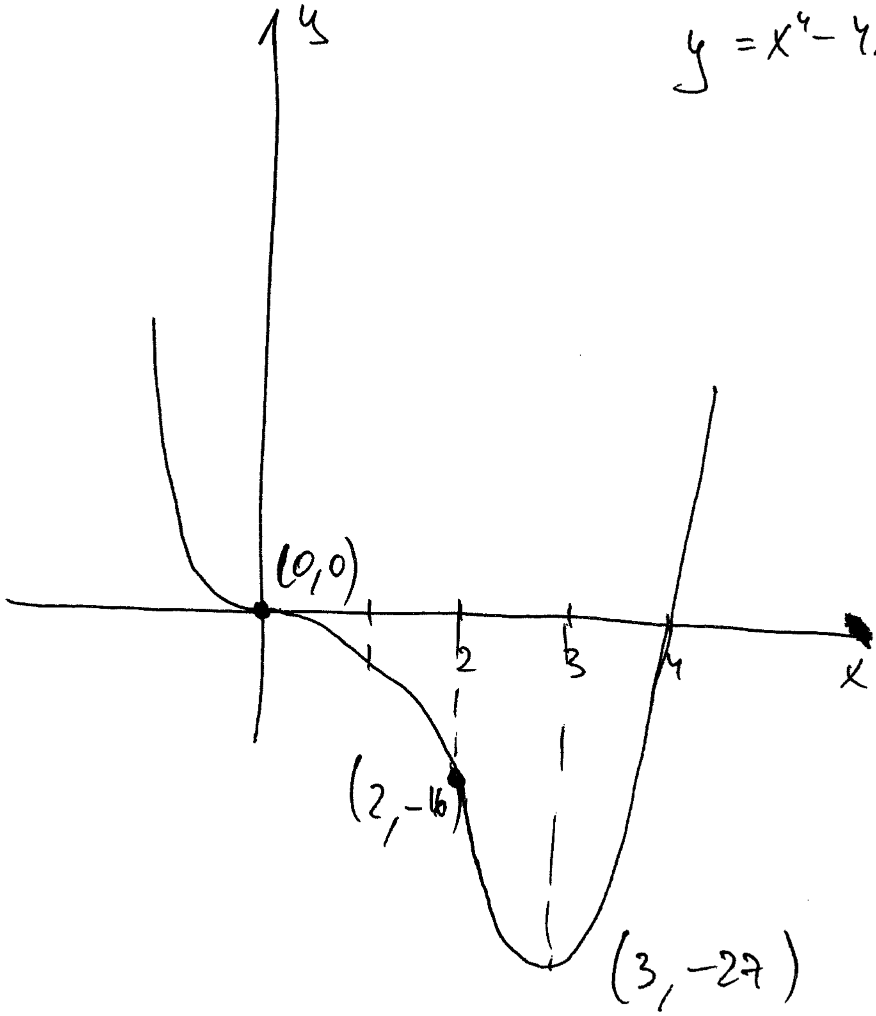
so f' doesn't change its sign meaning that f does not have a local max/min at 0.

Since $f''(x) = 0$ for $x = 0, 2$

Interval	$f''(x) = 12x(x-2)$	Concativity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

The point $(0,0)$ is an inflection point, since the curve changes from c.upward to c.downward. Also $(2, -16)$

$y = x^4 - 4x^3$



Indeterminate forms and l'Hospital's Rule

The limit $\lim_{x \to a} \frac{f(x)}{g(x)}$, where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.

is called an indeterminate form of type $\frac{0}{0}$

Example $\lim_{x \to 1} \frac{\ln(x)}{x-1}$

The limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ where $f(x) \to \infty$ $g(x) \to \infty$ is called an indeterminate form of type $\frac{\infty}{\infty}$

Example $\lim_{x \to \infty} \frac{\ln(x)}{x-1}$

L'Hospital's Rule

(13-11)

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a)

Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Here $x \to a$ can be replaced by $x \to a^+$ or $x \to a^-$, $x \to \infty$ or $x \to -\infty$

Find

(13-12)

$$\lim_{x \to 1} \frac{\ln(x)}{x-1}$$

$$\lim_{x \to 1} \ln(x) = \ln 1 = 0 \quad \text{and} \quad \lim_{x \to 1} (x-1) = 0$$

So $$\lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = 1.$$

Indeterminate Products

$\lim_{x \to a} f(x) \cdot g(x)$ where ~~lim~~ $f(x) \to 0$ and $g(x) \to \pm\infty$

— " — of type $0 \cdot \infty$

Differences

— " — of type $\infty - \infty$.

Powers

$\lim_{x \to a} f(x)^{g(x)}$ $\quad 0^0$, ∞^0, 1^∞

Take the logarithm.

Indeterminate Products

(14-1)

$\lim_{x\to a} f(x)\cdot g(x)$ where $f(x)\to 0$ and $g(x)\to\pm\infty$

indeterminate form of type $0\cdot\infty$

$$f\cdot g = \frac{f}{1/g} \quad \text{or} \quad f\cdot g = \frac{g}{1/f}$$

<u>Example</u> $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{L.R.}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} =$

$= \lim_{x\to 0^+} (-x) = 0.$

Indeterminate differences

(14-2)

$\lim_{x\to a} (f(x)-g(x))$ where $f(x)\to\infty$ and $g(x)\to\infty$

indeterminate form of type $\infty-\infty$

<u>Example</u> $\lim_{x\to(\frac{\pi}{2})^-} (\sec(x)-\tan(x)) = \lim_{x\to(\frac{\pi}{2})^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) =$

$= \lim_{x\to(\frac{\pi}{2})^-} \frac{1-\sin(x)}{\cos(x)} \overset{L.R.}{=} \lim_{x\to(\frac{\pi}{2})^-} \frac{-\cos x}{-\sin x} = 0$

Indeterminate Powers. (14-3)

$$\lim_{x \to a} f(x)^{g(x)}$$

1. $f(x) \to 0$ and $g(x) \to 0$
   type 0^0
2. $f(x) \to \infty$ and $g(x) \to 0$
   type ∞^0
3. $f(x) \to 1$ and $g(x) \to \pm\infty$
   type 1^∞

let $y = f(x)^{g(x)}$

$\ln y = g(x) \ln f(x)$

or

$f(x)^{g(x)} = e^{g(x) \ln f(x)}$

$g(x) \ln f(x)$ has type $0 \cdot \infty$.

Example Find $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$ (14-4)

type 1^∞

$y = (1 + \sin 4x)^{\cot x}$

$\ln y = \cot x \cdot \ln(1 + \sin 4x)$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} \overset{L.R.}{=} \lim_{x \to 0^+} \frac{\frac{4\cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4.$$

Optimization problems

<u>Minimizing cost</u> A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

We want to minimize the total surface area of the cylinder

$$A = 2\pi r^2 + 2\pi r h \qquad (r \text{ radius},\ h \text{ height}).$$

A is the total surface.

Volume is $\pi r^2 \cdot h = 1000\ cm^3$, so $h = \dfrac{1000}{\pi r^2}$

Substituting we obtain

$$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}, \quad r > 0$$

Find critical numbers

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

$A'(r) = 0$ when $\pi r^3 = 500$, so the only critical value is $r = \sqrt[3]{500/\pi}$

Note that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$

so $r = \sqrt[3]{500/\pi}$ gives an absolute minimum of $A(r)$

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

$r = \sqrt[3]{500/\pi}$ and $h = 2r$.

First Derivative Test for Absolute Extreme Values

(14-7)

Suppose that c is a critical number of a continuous function f defined on an interval

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example. Implicit differentiation.

(14-8)

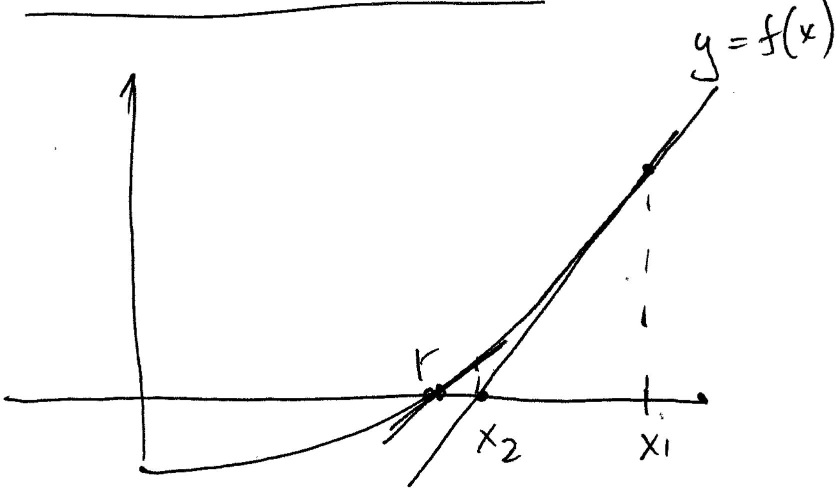
$$A(r) = 2\pi r^2 + 2\pi r h, \quad \pi r^2 h = 1000$$

$$A' = 4\pi r + 2\pi h + 2\pi r h' \qquad 2\pi r h + \pi r^2 h' = 0$$

Set $A' = 0$, then $2r + h + rh' = 0$ and $2h + rh' = 0$

So $2r - h = 0$ or $\underline{h = 2r}$

Newton's method



formula for x_2:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

slope of the tangent line at $(x_1, f$

Set $x = x_2$, $y = 0$, then

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

so $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Repeat:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad \text{So} \quad \lim_{n\to\infty} x_n = r$$

14-10

Example Starting with $x_1 = 2$ find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$

$$f(x) = x^3 - 2x - 5 \qquad f'(x) = 3x^2 - 2$$

$x_1 = 2$

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}.$$

$x_2 = 2.1$

$x_3 \approx 2.0946$.

Suppose that we want to achieve a given accuracy, then we can stop when successive approximations x_n and x_{n+1} agree up to that accuracy.

Example Starting with $x_1=2$ find the third approximation x_3 to the root of the equation $x^3-2x-5=0$ (15-1)

$f(x)=x^3-2x-5 \qquad f'(x)=3x^2-2$

$\underline{x_1=2} \qquad x_{n+1}=x_n-\dfrac{x_n^3-2x_n-5}{3x_n^2-2}.$

$x_2 = 2.1$

$x_3 \approx 2.0946.$

Suppose that we want to achieve a given accuracy, then we can stop when successive approximations x_n and x_{n+1} agree up to that accuracy.

Antiderivatives (15-2)

To find a function F whose derivative is a known function f. If such a function exists, it is called an anti-derivative of f

~~Defn~~ that is $F'(x)=f(x)$ for all x in the interval I

If F is an antiderivative of f on an interval I then the most general antiderivative of f on I is $F(x)+C$, where C is an arbitrary constant

Example Find the most general antiderivative of the following function

(a) $f(x) = \sin(x)$

If $F(x) = -\cos(x)$, then $F'(x) = \sin(x)$

So the most general antiderivative is

$G(x) = -\cos(x) + C$

(b) $f(x) = \frac{1}{x}$

$F(x) = \ln(x)$ and $F'(x) = \frac{1}{x}$ on the interval $(0, \infty)$

$F(x) = \ln(-x)$ and $F'(x) = -\frac{1}{x}$ on the interval $(-\infty, 0)$

So $G(x) = \begin{cases} \ln(x) + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0. \end{cases}$

(c) $f(x) = x^n$, $n \geq 0$.

$F(x) = \frac{x^{n+1}}{n+1}$ so $F'(x) = x^n$

and $G(x) = F(x) + C = \frac{x^{n+1}}{n+1} + C$

Table of antiderivatives

Function	antiderivative
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$

Example Find all functions g such that $g'(x) = 4\sin(x) + \dfrac{2x^5 - \sqrt{x}}{x}$ (15-5)

We rewrite:

$$g'(x) = 4\sin(x) + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin(x) + 2x^4 - \frac{1}{\sqrt{x}}$$

Using formulas

$$g(x) = -4\cos(x) + 2\cdot\frac{x^5}{5} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= -4\cos(x) + \frac{2}{5}x^5 - 2\sqrt{x} + C$$

Example Find f if $f'(x) = e^x + 20\left(1+x^2\right)^{-1}$ (15-6)
and $f(0) = -2$

First we find the general antiderivative

$$f'(x) = e^x + \frac{20}{1+x^2}$$

$$f(x) = e^x + 20\cdot\arctan(x) + C$$

To determine C we use the fact that $f(0) = -2$

$$f(0) = \underbrace{e^0}_{=1} + 20\,\underbrace{\arctan(0)}_{=0} + C = -2$$

So $C = -1-2 = -3$

$$f(x) = e^x + 20\cdot\arctan(x) - 3$$

Example Find f if $f''(x) = 12x^2 + 6x - 4$
$f(0) = 4$ and $f(1) = 1$

The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^2}{2} - 4x + C =$$
$$= 4x^3 + 3x^2 - 4x + C$$

$$f(x) = 4 \cdot \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + Cx + D =$$
$$= x^4 + x^3 - 2x^2 + Cx + D$$

$f(0) = D = 4$ and $f(1) = 1 + 1 - 2 + C + D = C + D = C + 4 = 1$

So $D = 4$

So $C = -3$

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

Example $s = f(t)$ position of an object
$v(t) = s'(t)$ velocity of the object

position is the antiderivative of the velocity

$a(t) = v'(t) = s''(t)$ acceleration of the object

velocity is the antiderivative of the acceleration.

If the acceleration and the initial values $s(0)$ and $v(0)$ are known, the the position can be found by antidifferentiating twice.

Example A particle moves in a straight line and has acceleration given by (15-9)

$$a(t) = 6t + 4$$

Its initial velocity is $v(0) = -6$ cm/s and its initial position is $s(0) = 9$cm

Find its position function $s(t)$

$$v'(t) = a(t) = 6t + 4$$

$$v(t) = 6 \cdot \frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

$$v(0) = C = -6$$

$$v(t) = 3t^2 + 4t - 6$$

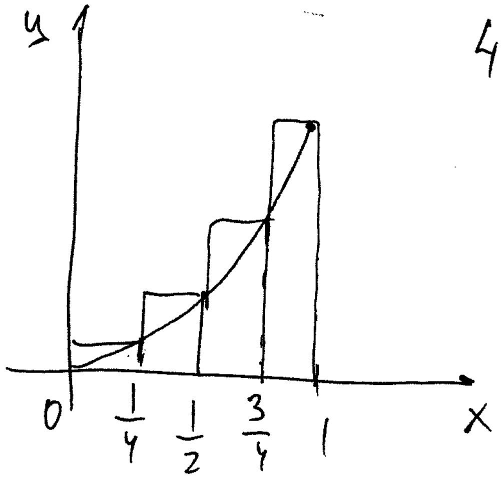
$$s(t) = 3 \cdot \frac{t^3}{3} + 4 \cdot \frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

so $s(0) = D = 9$

$$s(t) = t^3 + 2t^2 - 6t + 9$$

The area problem

Estimate the area under the parabola $y = x^2$ from 0 to 1



4 Strips $\frac{1}{4}$ width.

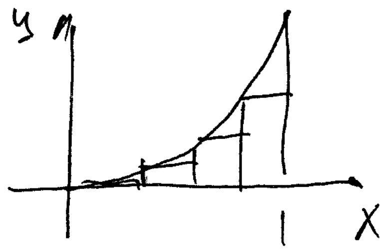
heights are $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2$, and 1^2

So

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 =$$

$$= \frac{15}{32} \approx 0.46875$$

Similarly,



$L_4 \approx 0.21875$

So $L_4 \leq A \leq R_4$

In general

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2 =$$

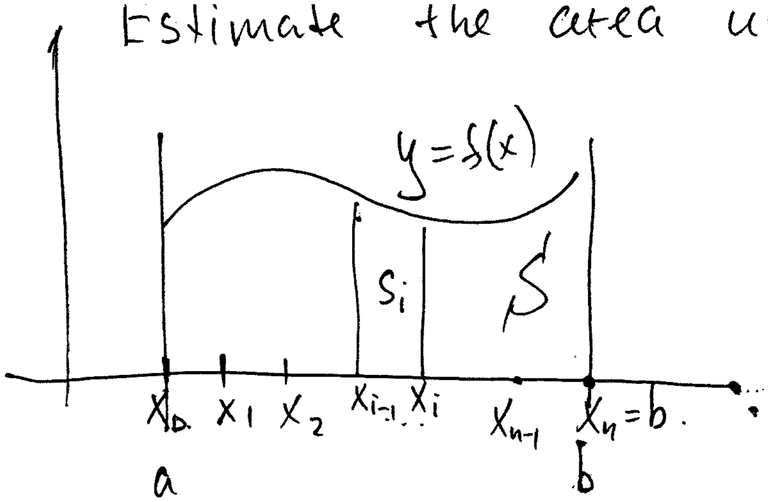
$$= \frac{1}{n^3}\left(1^2 + 2^2 + \ldots + n^2\right) = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

So $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}$

Hence, $A = \frac{1}{3}$

Observe that $\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \frac{1}{3} = A$.

Estimate the area under $y = f(x)$: (16-3)



$\Delta x = \dfrac{b-a}{n}$ the width of each of the n strips

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\vdots$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

The area A of the region S that lies under the graph of the continuous function f is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n, \text{ where } L_n = f(x_0)\Delta x + \dots + f(x_{n-1})\Delta x$$

In fact, instead of using left or right endpoints, we could take the height of the i-th rectangle to be the value of f at any number x_i^* in the i-th subinterval $[x_{i-1}, x_i]$. (16-4)

We call $x_1^*, x_2^*, \dots, x_n^*$ the <u>sample</u> points

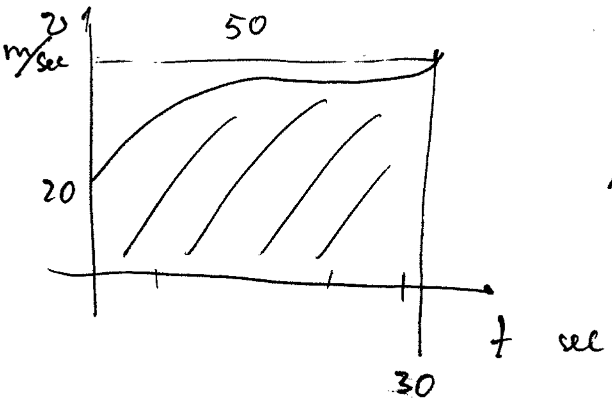
So $$A = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\right) \cdot \Delta x$$

The distance problem

16-5

distance = velocity × time

What if velocity varies?



A = distance.

The Definite Integral

16-6

If f is a function defined for $a \le x \le b$ we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.

Let $x_0 = a, \ldots, x_n = b$ be endpoints

Let $x_1^*, \ldots, x_n^*$ be sample points, so $x_i^* \in [x_{i-1}, x_i]$

Then $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists.

If it does exist, we say that f is integrable on $[a,b]$

16-7

$\int$, integral sign $\qquad \int_a^b$ limits of integration

$f(x)$ integrand $\qquad a$ = lower limit
$\qquad\qquad\qquad\qquad b$ = upper limit

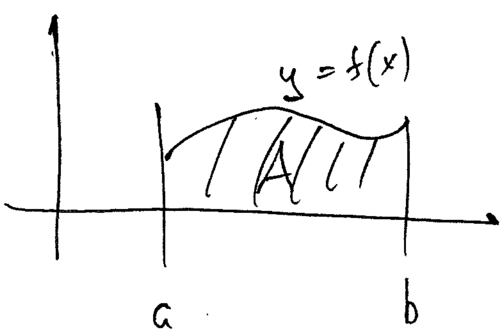
The procedure of calculating an integral is called integration.

$\int_a^b f(x)dx$ is a number!!! it does not depend on x.

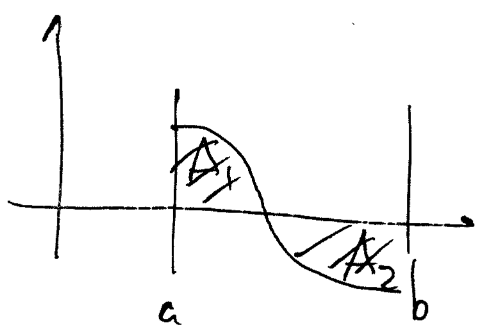
$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

$\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum.

16-8



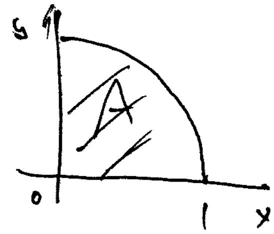
$$A = \int_a^b f(x)\,dx$$



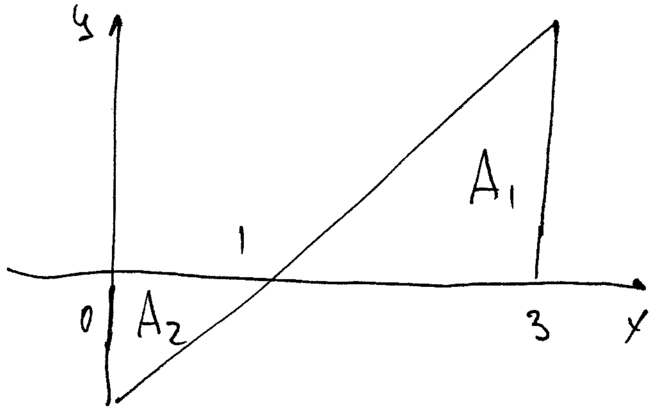
$$\int_a^b f(x)dx = A_1 - A_2.$$

If f is continuous on $[a,b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$.

Example $A = \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4} \cdot \pi (1)^2 = \frac{\pi}{4}$. (16-9)



$$\int_0^3 (x-1)\,dx = A_1 - A_2 =$$
$$= \frac{1}{2}(2 \cdot 2) - \frac{1}{2}(1 \cdot 1) = 1.5.$$



The Midpoint Rule (16-10)

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x \left(f(\bar{x}_1) + \ldots + f(\bar{x}_n) \right)$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$

$=$ midpoint of $[x_{i-1}, x_i]$

Example Use the Midpoint Rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$

Endpoints $1,\ 1.2,\ 1.4,\ 1.6,\ 1.8$ and 2.0

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x \left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right] =$$
$$= \frac{1}{5} \left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right) \approx 0.69190$$

Properties

(16-11)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

1. $\int_a^b c\,dx = c(b-a)$ C is a constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b c f(x)\,dx = c \cdot \int_a^b f(x)\,dx$
4. $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Comparison properties

(16-12)

6. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Example Use the properties of integral to evaluate $\int_0^1 (4+3x^2)dx$

We have $\int_0^1 (4+3x^2)dx = \int_0^1 4dx + \int_0^1 3x^2dx =$

$= \int_0^1 4dx + 3\int_0^1 x^2dx = 4(1-0) + 3\cdot\frac{1}{3} = 5.$

Example Use property 8 to estimate $\int_0^1 e^{-x^2}dx$

$f(x) = e^{-x^2}$ is a decreasing function on $[0,1]$

absolute maximum is $f(0)=1$

absolute minimum is $f(1) = e^{-1}$

So $e^{-1}(1-0) \le \int_0^1 e^{-x^2}dx \le 1(1-0)$

$e^{-1} \le \int_0^1 e^{-x^2}dx \le 1$, $e^{-1} = 0.367$

Evaluation Theorem

If f is continuous on the interval $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f, that is $F' = f$.

---

divide $[a,b]$ into n subintervals

end points $x_0 = a, x_1, \ldots, x_n = b$

$F(b) - F(a) = \underbrace{F(x_n)}_{F(b)} - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \ldots + F(x_1) - \underbrace{F(x_0)}_{F(a)}$

By the Mean Value Theorem $F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) =$

$= f(x_i^*)(x_i - x_{i-1}) =$

$= f(x_i^*)\cdot\Delta x$

So $F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$ and

Find $\int_1^3 e^x dx = e^x\Big|_1^3 = e^3 - e^1$ (17-3)

The notation $\int f(x)dx$ is used for a general antiderivative of f and is called an indefinite integral.

Indefinite integral is a function!

$$\int_a^b f(x)dx = \left(\int f(x)dx\right)\Big|_a^b$$

Example $\int \frac{1}{x}dx = \ln|x| + C$

One can use the table of antiderivatives to find the integr

Properties (17-4)

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

$$\int cf(x)\,dx = c\int f(x)dx \quad \text{for any constant } c$$

Example Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)dx = 10\int x^4 dx - 2\int \sec^2 x dx =$$

$$= 10\,\frac{x^5}{5} - 2\tan x + C =$$

$$= 2x^5 - 2\tan x + C$$

Example Evaluate $\int_1^9 \frac{2t^2+t^2\sqrt{t}-1}{t^2}\,dt =$ (17-5)

$$= \int_1^9 (2+t^{\frac{1}{2}}-t^{-2})\,dt = 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Big|_1^9 =$$

$$= 2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}\Big|_1^9 = \left[2\cdot 9 + \frac{2}{3}(9)^{\frac{3}{2}} + \frac{1}{9}\right] -$$

$$- \left[2\cdot 1 + \frac{2}{3}\cdot 1^{3/2} + \frac{1}{1}\right] = 32\tfrac{4}{9}$$

(17-6)

Net Change Theorem

The integral of a rate of change is the net change

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example If $C(t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $C'(t)$

So $\int_{t_1}^{t_2} C'(t)\,dt = C(t_2) - C(t_1)$ is the change in the concentration of C from time t_1 to t_2.

Example A particle moves along a line so that its velocity at time t is (17-7)

$v(t) = t^2 - t - 6$ m/sec

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$

(b) Find the distance traveled during this time period.

(a): $$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt = \left.\frac{t^3}{3} - \frac{t^2}{2} - 6t\right|_1^4 = -\frac{9}{2}.$$

The particle position at $t=4$ is 4.5m to the left of its position at $t=1$.

(b): $v(t) = t^2 - t - 6 = (t-3)(t+2)$ (17-8)

so $v(t) \leq 0$ on $[1,3]$ and

$v(t) \geq 0$ on $[3,4]$

$$\int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt =$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt =$$

$\approx$ 10.17 m.

Let f be a continuous function on $[a,b]$ and $a \le x \le b$

Consider $\int_a^x f(t)\,dt =: g(x)$

Example If $g(x) = \int_a^x f(t)\,dt$ where $a=1$ and $f(t)=t^2$

Find a formula for $g(x)$ and for $g'(x)$

$$g(x) = \int_1^x t^2\,dt = \frac{t^3}{3}\Big|_1^x = \frac{x^3}{3} - \frac{1}{3} = \frac{x^3-1}{3}$$

$$g'(x) = x^2.$$

The Fundamental Theorem of Calculus

1. If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \quad , \quad a \le x \le b$$

is an antiderivative of f, that is $g'(x) = f(x)$ for $a < x < b$.

2. The evaluation theorem

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big]_a^b$$

Example Find the derivative of (18-3)

$$g(x) = \int_0^x \sqrt{1+t^2}\,dt$$

Answer $g'(x) = \sqrt{1+x^2}$

Example Find $\left(\int_1^{x^4} \sec(t)\,dt\right)' = \dfrac{d \int_1^{x^4} \sec(t)\,dt}{dx} =$

Let $u = x^4$. Then

$$= \frac{d}{du}\left(\int_1^u \sec t\,dt\right) \cdot \frac{du}{dx} = \sec u \cdot \frac{du}{dx} = \sec x^4 \cdot 4x^3.$$

The Substitution Rule $u = 1+x^2$ (18-4)

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\;2x\,dx = \int \sqrt{u}\,du =$$

$(du = u'(x)\,dx = 2x\,dx)$

$$= \frac{2}{3}u^{\frac{3}{2}} + C =$$

$$= \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$$

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$$

or $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$ is a differentiable function on an interva

Example   Find $\int x^3 \cos(x^4+2)\,dx$

$u = x^4+2 \qquad u' = 4x^3$ so

$$= \int \cos u \cdot \tfrac{1}{4}du = \tfrac{1}{4}\int \cos u\, du = \tfrac{1}{4}\sin u + C =$$

$$= \tfrac{1}{4}\sin(x^4+2) + C$$

Find $\int \frac{x}{\sqrt{1-4x^2}}dx$: Let $u = 1-4x^2$. Then $u' = -8x$

so

$$= -\frac{1}{8}\int \frac{1}{\sqrt{u}}\cdot(-8x)\,dx = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\,2\sqrt{u} + C =$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = \int \frac{1}{\cos x}(\sin x\, dx) =$$

$u = \cos x$

$$= -\int \frac{1}{u}du = -\ln|u| + C = -\ln|\cos x| + C$$

The substitution rule for definite integrals

If g' is continuous on $[a,b]$ and f is continuous on the range of $u=g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

(18-7)

Find $\displaystyle\int_0^4 \sqrt{2x+1}\,dx =$ $\quad u = 2x+1$

$du = 2\cdot dx \quad$ so

$$= \frac{1}{2}\int_1^9 u^{\frac{1}{2}}\,du = \frac{1}{2}\cdot \frac{2}{3}u^{\frac{3}{2}}\Big]_1^9 = \frac{1}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{26}{3}$$

$1 = 2\cdot 0+1$
$9 = 2\cdot 4+1$

$u = \ln x \qquad du = \frac{1}{x}\cdot dx$

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}.$$

(18-8)

Integrals of symmetric functions

Suppose f is continuous on $[-a,a]$

(a) If f is even then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) If f is odd then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

Indeed, $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$

Examples $\displaystyle\int_{-2}^{2} (x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx = 2\left(\frac{1}{7}x^7 + x\right)\Big]_0^2 = \frac{284}{7}$

Integration by parts

Product Rule $(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$

$$\int (f(x)g'(x) + f'(x)g(x))dx = \int (f(x)g(x))'dx = f(x)g(x)$$

or $\int f(x)g'(x)dx + \int f'(x)g(x)dx = f(x)g(x)$

or $\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx}$

is called the formula for integration by parts

Let $u = f(x)$ and $v = g(x)$

Then $du = f'(x)dx$ and $dv = g'(x)dx$

So by the substitution rule

$$\int u\,dv = u \cdot v - \int v\,du$$

Example Find $\int x \sin x\, dx$

$f(x) = x$ and $g'(x) = \sin x$

Then $f'(x) = 1$ and $g(x) = -\cos x$ (choose any antiderivative of g

$\int x \sin x\, dx = x(-\cos x) - \int (-\cos x)dx =$

$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C.$

(19-3)

Find $\int \ln x \, dx$

Let $f(x) = \ln x$ and $g(x) = x$

Then $f'(x) = \frac{1}{x}$ and $g'(x) = 1$.

So $\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$

$\int t^2 e^t dt = ?$ (The derivative of f has to simpler than f)

$f(t) = t^2$ $g(t) = e^t$ so $g'(t) = e^t$

$f'(t) = 2t$

$\int t^2 e^t \, dt = t^2 \cdot e^t - \int 2t e^t dt = t^2 \cdot e^t - 2 \int t e^t dt$

Again $f(t) = t$ $g(t) = e^t$ so $f'(t) = 1$

$\int t e^t dt = t \cdot e^t - \int e^t dt = t \cdot e^t - e^t + C$.

(19-4)

$\int e^x \sin x \, dx$

$f(x) = e^x$ $g(x) = -\cos x$ $g'(x) = \sin x$ $f'(x) = e^x$

$\int e^x \sin x \, dx = -e^x \cos x - \int e^x \cdot (-\cos x) dx =$

$= -e^x \cos x + \int e^x \cos x \, dx$

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

$f = e^x$ $g = \sin x$

$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$

So $2 \cdot \int e^x \sin x \, dx = e^x (\sin x - \cos x)$

So $\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$.

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)dx$$ (19-5)

<u>Example</u> $\int_0^1 \arctan x \, dx$

Let $f(x) = \arctan(x)$ $g(x) = x$ $f'(x) = \frac{1}{1+x^2}$ $g'(x) = 1$

$$\int_0^1 \arctan x \, dx = x\arctan(x)\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}dx =$$

$$= 1\cdot\arctan(1) - 0\cdot\arctan(0) - \int_0^1 \frac{x}{1+x^2}dx =$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx.$$

(19-6)

$\int_0^1 \frac{x}{1+x^2}dx =$ use substitution $u = 1+x^2$ $u' = 2x$

$du = 2x\,dx$

$\cancel{u=}$

So $= \frac{1}{2}\int_0^1 \frac{2x\,dx}{1+x^2} = \frac{1}{2}\int_{u(0)}^{u(1)} \frac{du}{u} = \frac{1}{2}\int_1^2 \frac{du}{u} = \frac{1}{2}\ln u\Big]_1^2 =$

$= \frac{1}{2}\ln 2 - \ln 1 = \frac{1}{2}\ln 2$

So $\int_0^1 \arctan(x)\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$

$\int x^2 \ln x \, dx$ $u = \ln x$ $dv = x^2 dx$

$\int \theta\cdot\cos\theta \, d\theta$ $u = \theta$ $dv = \cos\theta \, d\theta.$

Trigonometric integrals

An integral with an odd power of cos x

$$\int \cos^3 x \, dx$$

We separate one cos-factor and convert the remaining $\cos^2 x$-factor to an expression involving sin:

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x\,(1-\sin^2 x)$$

$$\int \cos^3 x\, dx = \int \cos x\,(1-\sin^2 x)\,dx =$$

Substitute
$u = \sin x$
$du = \cos x\, dx$

$$= \int (1-u^2)\,du = u - \frac{u^3}{3} + C =$$

$$= \sin x - \frac{\sin^3 x}{3} + C.$$

An integral with an even power of sin x.

$$\int_0^{\pi} \sin^2 x\, dx =$$ Use half-angle formula

$$= \frac{1}{2}\int_0^{\pi} (1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big]_0^{\pi} =$$

$$= \frac{1}{2}\left(\pi - \frac{1}{2}\sin 2\pi\right) - \frac{1}{2}\left(0 - \frac{1}{2}\sin 0\right) = \frac{1}{2}\pi$$

Partial Fractions (20-3)

$$\int \frac{5x-4}{2x^2+x-1}\,dx \;\circleddash\qquad \frac{5x-4}{2x^2+x-1} = \frac{5x-4}{(x+1)(2x-1)} =$$

$$= \frac{A}{x+1} + \frac{B}{2x-1}$$ Find A and B:

$$5x-4 = A(2x-1) + B(x+1)$$
$$5x-4 = (2A+B)x + (-A+B)$$

$$\begin{cases} 2A+B=5 \\ -A+B=-4 \end{cases}$$ So $A=3$ and $B=-1$

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{3}{x+1} - \frac{1}{2x-1}$$

(20-4)

$$\int \frac{5x-4}{2x^2+x-1}\,dx = \int\left(\frac{3}{x+1} - \frac{1}{2x-1}\right)dx =$$

$$= \int \frac{3}{x+1}\,dx - \int \frac{1}{2x-1}\,dx = 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + C$$

---

1. ~~The~~ We have to make the degree of the numerator strictly less that the degree of the denominator.
2. If the denominator has more than two linear factors, we have to include a term corresponding to each factor

$$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}$$

(20-5)

3. If a linear factor is repeated, we have to include extra terms for each intermediate power:

$$\frac{x}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}$$

4. If we obtain an irreducible quadratic factor ax^2+bx+c in the denominator, then we put $\dfrac{Ax+B}{ax^2+bx+c}$ and find A, B.

This term can be integrated by completing the square and using $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$.

(20-6)

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx =$$

$$x^3+4x = x(x^2+4)$$

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4\cdot A$$

$$\begin{cases} A+B=2 \\ C=-1 \\ 4A=4 \end{cases} \qquad \text{So } A=1,\ B=1,\ C=-1$$

$$= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx =$$

$\int \frac{x}{x^2+4} dx = \quad (u = x^2+4 \quad du = 2x dx$ 20-7

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2} \ln(u) + C = \frac{1}{2} \ln(x^2+4) + C$

$\int \frac{1}{x^2+4} dx = \frac{1}{2} \arctan\left(\frac{x}{2}\right) + D$

$\int \frac{1}{x} dx = \ln|x| + E$ $\quad (K = C + D + E$

So $= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K$

$\int \sin^3 x \cos^2 dx$

$\int_0^{\pi/2} \cos^5 x \, dx$

(21-1)

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2+x+2+\frac{2}{x-1}\right)dx =$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx \ \circledequal$$

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1) =$$

$$= (2A+B+2C)x^2+(3A+2B-C)x-2A$$

(21-2)

$$\begin{cases} 2A+B+2C=1 \\ 3A+2B-C=2 \\ -2A=-1 \end{cases} \qquad A=\frac{1}{2} \qquad B=\frac{1}{5} \qquad C=-\frac{1}{10}$$

So $\circledequal \int\left(\frac{1}{2}\cdot\frac{1}{x}+\frac{1}{5}\cdot\frac{1}{2x-1}-\frac{1}{10}\cdot\frac{1}{x+2}\right)dx =$

$$= \frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K.$$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx =$$ (21-3)

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}$$

$$x^3-x^2-x+1 = (x-1)(x^2-1) = (x-1)^2(x+1)$$

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 =$$

$$= (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$\begin{cases} A+C=0 \\ B-2C=4 \\ -A+B+C=0 \end{cases} \Rightarrow A=1,\ B=2,\ C=-1$$

(21-4)

So $\cancel{\int} = \int\left(x+1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right)dx =$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K =$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K.$$

$$\cancel{\int \frac{2x^2-x+4}{x^3+4x}\,dx = ?}$$

Approximate Integration

$$\int_a^b f(x)\,dx \approx L_n \quad \text{left-endpoint approximation}$$

$$\int_a^b f(x)\,dx \approx R_n \quad \text{Right-endpoint approximation}$$

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1) + \dots + f(\bar{x}_n)\right]$$

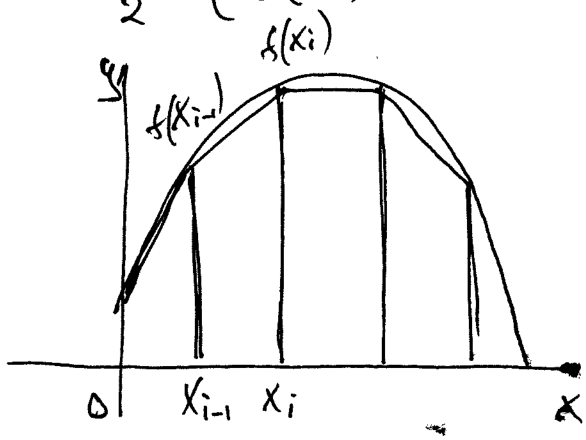
$$\Delta x = \frac{b-a}{n} \qquad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$$

Midpoint approximation.

$$\int_a^b f(x)\,dx \approx \frac{L_n + R_n}{2} \quad \text{Trapezoidal approximation.}$$

$$T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}\left[\sum_{i=1}^{n} f(x_{i-1}) + f(x_i)\right] =$$

$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$



Example Use the Trapezoidal Rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$

$a=1$, $b=2$ $\quad \Delta x = \frac{b-a}{5} = \frac{1}{5} = 0.2.$

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{0.2}{2}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right) \approx 0.6956$$

Error $\quad E_B := \int_a^b f(x)\,dx - B_n$

$B_n = L_n, R_n, M_n$ or T_n

Error Bounds Suppose $|f''(x)| \le K$ for $a \le x \le b$

$$\cancel{E}\ |E_T| \le \frac{K(b-a)^2}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

if $f(x) = \frac{1}{x}$ $\quad f''(x) = \frac{2}{x^3}$

Since $1 \le x \le 2$ $\quad \frac{1}{x} \le 1$, so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2 \Rightarrow |E_T| \le \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx$$

≈ 0.00666