THE UNIVERSITY OF BRITISH COLUMBIA

MATH 104 (106)

Midterm 2

3 November 2009

TIME: 80 MINUTES

FULL NAME: Solutions STUDENT # :

SIGNATURE:

This Examination paper consists of 9 pages (including this one). Make sure you have all 9.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1	/10
Q2	/15
Q3	/10
Q4	/5
Q5	/10
TOTAL	/50

NAME OF INSTRUCTOR: Jim Bryan

Q1 [10 marks]

Find the derivatives of the following functions. Select and clearly circle the correct answer from the multiple choice. No partial credit will be given.

- $e^{7x} + 3e^{2x} + 2e^{x}$

  (a) $e^{7x} + 3e^{2x} + 2e^{x}$
  (b) $7xe^{7x-1} + 6xe^{2x-1} + 2xe^{x-1}$
  (c) $7xe^{7x} + 6xe^{2x} + 2xe^{x}$
  (d) $7e^{7x} + 6e^{2x} + 2e^{x}$
  (e) None of the above

- $\ln(\ln(x))$

  (a) $\dfrac{1}{x^2}$
  (b) $\dfrac{1}{\ln(x)}$
  (c) $\dfrac{1}{x\ln(x)}$
  (d) $\dfrac{2\ln(x)}{x}$
  (e) None of the above.

- $e^{x}x^{e}$

  (a) $e^{x}x^{e-1}(e+x)$
  (b) $e^{x+1}x^{e-1}$
  (c) $2e^{x}x^{e}$
  (d) $(xe^{x-1})(ex^{e-1})$
  (e) None of the above.

$$\frac{d}{dx}(e^x x^e) = e^x(e x^{e-1}) + e^x x^e$$
$$= e^{x+1} x^{e-1} + e^x x^e$$
$$= e^x x^{e-1}(e + x)$$

- $\ln(x) - \ln(\frac{1}{x}) - \ln(x^2)$ $= \ln(x) + \ln(x) - 2\ln(x) = 0$

or taking derivatives term by term

$$\frac{1}{x} - \left(\frac{1}{x}\right)^{-1}\left(\frac{-1}{x^2}\right) - \frac{1}{x^2}(2x)$$

$$= \frac{1}{x} + \frac{x}{x^2} - \frac{2x}{x^2} = \frac{1}{x} + \frac{1}{x} - \frac{2}{x} = 0$$

(a) $\frac{1}{x} - x - \frac{1}{x^2}$

(b) $\frac{1}{x} - \frac{x}{x^2} - \frac{2x}{x^2}$

(c) $\frac{2}{x}$

(d) 0

(e) None of the above.

- $\dfrac{\ln(x) + x^{1/5}}{(1+e^{3x})^2}$

(a) $\dfrac{\frac{1}{x}\left(1+\frac{1}{5}x^{1/5}\right)(1+e^{3x}) - 6e^{3x}\left(\ln(x)+x^{1/5}\right)}{(1+e^{3x})^3}$

(b) $\dfrac{\frac{1}{x}+\frac{1}{5}x^{-4/5}}{2(1+e^{3x})}$

(c) $\dfrac{\left(\frac{1}{x}+\frac{1}{5}x^{-4/5}\right)(1+e^{3x})^2 - 2(1+e^{3x})\left(\ln(x)+x^{1/5}\right)}{(1+e^{3x})^4}$

(d) $\dfrac{\left(\frac{1}{x}+\frac{1}{5}x^{-4/5}\right)(1+e^{3x})^2 - 2e^{3x}(1+e^{3x})\left(\ln(x)+x^{1/5}\right)}{(1+e^{3x})^4}$

(e) None of the above.

quotient rule:

$$\frac{\left(\frac{1}{x}+\frac{1}{5}x^{-4/5}\right)(1+e^{3x})^2 - 2(1+e^{3x})3e^{3x}\left(\ln(x)+x^{1/5}\right)}{(1+e^{3x})^4}$$

$$= \frac{\frac{1}{x}\left(1+\frac{1}{5}x^{1/5}\right)(1+e^{3x}) - 6e^{3x}\left(\ln(x)+x^{1/5}\right)}{(1+e^{3x})^3}$$

Q2 [15 marks]

Consider the function
$$f(x) = \frac{x^2}{4} + \frac{x}{2} - \frac{7}{4} + \frac{2}{x-2}.$$

(a) Find all x such that $f'(x) = 0$.

$f'(x) = \frac{x}{2} + \frac{1}{2} - \frac{2}{(x-2)^2} = 0$ $\quad \frac{1}{2}(x+1) = \frac{2}{(x-2)^2}$ $\quad (x-2)^2(x+1) = 4$

$\Rightarrow (x^2-4x+\cancel{4})(x+1) = 4 \Rightarrow x^3 - 4x^2 + \cancel{4x} + x^2 - \cancel{4x} + \cancel{4} = \cancel{4}$ $\quad x^3 - 3x^2 = 0$

$\boxed{x = 0 \text{ or } x = 3}$

(b) Find all x such that $f''(x) = 0$.

$f''(x) = \frac{1}{2} + \frac{4}{(x-2)^3} = 0 \Rightarrow (x-2)^3 = -8 \Rightarrow x-2 = -2$

$\Rightarrow \boxed{x = 0}$

(c) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?

$-\infty < x < 0$: $f'(-1) = -\frac{1}{2} + \frac{1}{2} - \frac{2}{(-3)^2} = -\frac{2}{9}$ neg : $\boxed{\text{decreasing}}$

$0 < x < 3$: $f'(1) = \frac{1}{2} + \frac{1}{2} - \frac{2}{(-1)^2} = 1 - 2 = -1$ neg : $\boxed{\text{decreasing}}$

$3 < x < \infty$: $f'(4) = \frac{4}{2} + \frac{1}{2} - \frac{2}{(4-2)^2} = \frac{5}{2} - \frac{2}{4} = 2$: $\boxed{\text{increasing}}$

(d) On which intervals is $f(x)$ concave up? On which intervals is $f(x)$ concave down?

$x < 0$ $\quad f''(-2) = \frac{1}{2} + \frac{4}{(-4)^3} = \frac{1}{2} - \frac{1}{16}$ positive $\boxed{\text{concave up}}$

$2 > x > 0$ $\quad f''(1) = \frac{1}{2} + \frac{4}{(-1)^3} = \frac{1}{2} - 4$ negative $\boxed{\text{concave down}}$

$2 < x$ $\quad f''(3) = \frac{1}{2} + \frac{4}{1}$ positive $\quad$ concave up

(e) For each critical point, determine whether it is a local maximum, a local minimum, or neither.

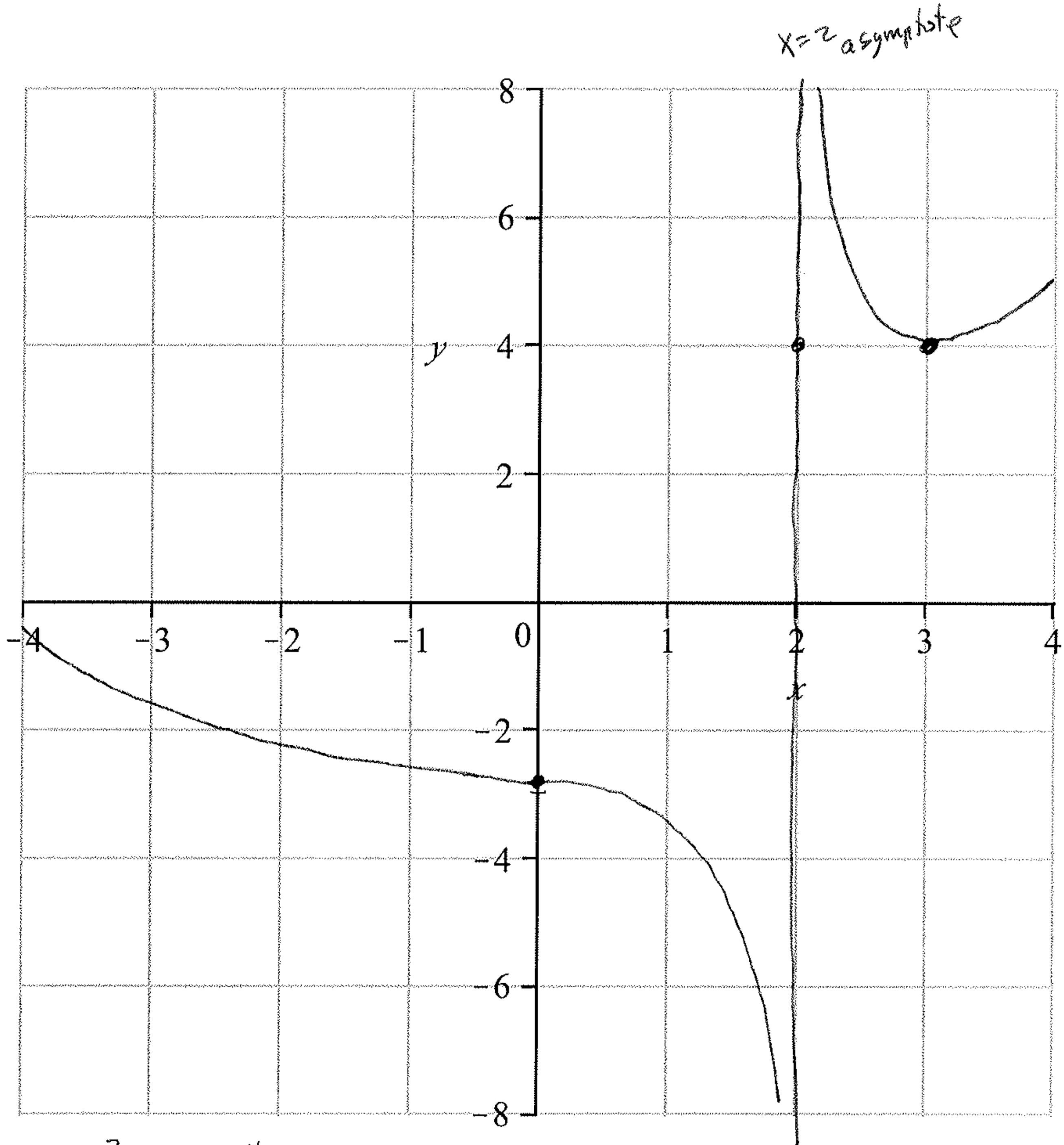
$x = 0$ $\quad f''(0) = 0$ $\quad \boxed{\text{neither local max or local min}}$

$x = 3$ $\quad f''(3) > 0$ $\quad \boxed{\text{local minimum}}$

(f) Does $f(x)$ have an asymptote? If so, write the equation for the asymptote and say whether it is a vertical, horizontal, or slant asymptote.

vertical asymptote given by $x = 2$

(g) Draw the graph of $f(x)$ on the graph provided. Acurately place all critical points and inflection points, indicate all asymptotes, and make sure your graph correctly shows where $f(x)$ is increasing and decreasing and correctly shows its concavity.



$x=0 \quad y = -\frac{7}{4} - 1 = -\frac{11}{4}$

$x=3 \quad y = \frac{9}{4} + \frac{3}{2} - \frac{7}{4} + 2 = \frac{1}{2} + \frac{3}{2} + 2 = 4$

Q3 [10 marks]

The Apple computer company wants to introduce its new iPod model the Fluffle. Market research shows that the demand for the old model, the Shuffle, will drop when the Fluffle becomes available. Apple estimates that it will sell 2 million Shuffles at the old price of \$50. To boost sales, Apple plans to offer a rebate on the Shuffles. Market research shows that at rebate of \$5 will boost sales by 500,000.

(*a*) Assuming a linear relationship between the rebate and sales, find a formula for $q(x)$, the number of millions of units sold when the rebate is x dollars.

$q = mx + b \qquad x = 0 \Rightarrow q = 2 \qquad \text{so} \quad b = 2$

$2 + \frac{1}{2} = m(5) + 2 \quad \Rightarrow \quad m = \frac{1}{10}$

$$\boxed{q = \frac{1}{10}x + 2}$$

(*b*) Find a formula for $R(x)$ the revenue from Shuffle sales, expressed in millions of dollars.

$\cancel{R(x) = xq = x(\tfrac{1}{10}x+2) = \tfrac{1}{10}x^2 + 2x}$

$R(x) = \text{price} \cdot q = (50 - x)q = (50 - x)(\frac{1}{10}x + 2)$

$= -\frac{x^2}{10} + 3x + 100$

(*c*) Find the rebate which will maximize revenue.

$R'(x) = -\frac{1}{5}x + 3 = 0 \qquad \boxed{x = 15}$

(*d*) The iPod shuffle costs Apple \$10 per unit to produce and the cost of advertizing the rebate is 2 million dollars. Find a formula for $P(x)$, the profit from Shuffle sales, expressed in millions of dollars.

$$\text{Cost} = 2 + 10g = 2 + 10\left(\frac{1}{10}x + 2\right) = x + 22$$

$$P(x) = R(x) - C(x) = -\frac{x^2}{10} + 3x + 100 - x - 22$$

$$\boxed{P(x) = -\frac{x^2}{10} + 2x + 88}$$

(*e*) Find the rebate which will maximize profit.

$$P'(x) = -\frac{x}{5} + 2 = 0 \qquad \boxed{x = 10}$$

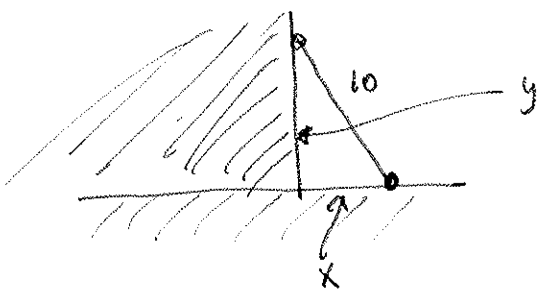
(*f*) How much profit (in millions of dollars) will apple make on the Shuffle sales?

$$P(10) = -\frac{100}{10} + 20 + 88 = 98 \text{ million dollars.}$$

Q4 [5 marks]

A 10 foot ladder is leaning against a wall. The foot of the ladder is on the floor at a point x feet from the wall and the top of the ladder is resting on the wall at a point y feet above the ground.

(a) Use the Pythagorean theorem to find an equation relating x and y.



$x^2+y^2=100$

(b) If the foot of the ladder is being pulled along the ground at the rate of 3 feet per second, how fast is the top of the ladder sliding down the wall at the time when the foot of the ladder is 8 feet from the wall?

$\frac{dx}{dt}=3$ what is $\frac{dy}{dt}$ at $x=8$?

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$ $x=8$ $y=6$

$2\cdot 8\cdot 3+2\cdot 6\cdot\frac{dy}{dt}=0$

$\frac{dy}{dt}=-\frac{2\cdot 8\cdot 3}{2\cdot 6}=-4$

Ladder is ~~takes~~ sliding down at 4 ft/sec

Q5 [10 marks]

(*a*) Find the equation of the line tangent to the graph $y = e^{-x}$ at $x = a$.

$y = mx + b \qquad m = \left.\frac{dy}{dx}\right|_{x=a} = -e^{-a}$

line passes through the point (a, e^{-a}).

$e^{-a} = -e^{-a} \cdot a + b \qquad b = e^{-a}(1+a)$

$$\boxed{y = -e^{-a}x + e^{-a}(1+a)}$$

(*b*) Assume that $a \geq 0$. Let A be the area of the triangle formed by the x-axis, the y-axis, and the tangent line (see the figure below). Show that A is given by

$$A = \frac{1}{2}(1+a)^2 e^{-a}$$

$A = \frac{1}{2}$ base $\times$ height $= \frac{1}{2}$(x-intercept)(y-intercept)

y-intercept $= e^{-a}(1+a) \qquad$ x-int: $0 = -e^{-a}x + e^{-a}(1+a) \Rightarrow x = (1+a)$

so $A = \frac{1}{2}(1+a)^2 e^{-a}$

(*c*) Find the value of a which maximizes the area of the triangle.

$A'(a) = (1+a)e^{-a} + \frac{1}{2}(1+a)^2(-e^{-a}) = 0$

$\cancel{e^{-a}}(1+a) = \frac{1}{2}(1+a)^2 \cancel{e^{-a}}$

$(1+a) = \frac{1}{2}(1+a)^2 \qquad a > 0$ so $1+a \neq 0$

$1 = \frac{1}{2}(1+a)$

$2 = 1+a \qquad \boxed{a = 1}$