Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional statement which is true because its hypothesis is false is said to be **vacuously true** or **true by default**.

Conditional Statements

Example

- If $1+1=2$, then $2+2=4$

  T

- If $1+1=2$, then $2+2=5$.

  F

- If $0=1$, then $2+2=4$.

  T

- If $0=1$, then $2+2=5$.

  T

Conditional Statements

Order of operations:

- brackets
- $\sim$
- $\wedge$, $\vee$
- $\rightarrow$

Example

$$\sim p \rightarrow q \vee r \equiv (\sim p) \rightarrow (q \vee r)$$

Conditional Statements

Example

Express the statement "If it is hot and it does not rain, then I will go to the park" symbolically.

Solution.

Let h: It is hot.
r: It rains.
p: I will go to the park.

The statement is

$$h \wedge \sim r \to p$$

Example

Construct a truth table for $\sim p \vee q \rightarrow p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \wedge \sim q$	$\sim p \vee q \rightarrow p \wedge \sim q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	F
F	F	T	T	T	F	F

Division into Cases

Show that $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns for $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ are identical, these statements are logically equivalent.

Representing $p \to q$ as an "or" statement

It can be shown (exercise) that

$$\boxed{p \to q \equiv \sim p \vee q}$$

Example

Rewrite the following statement in an if-then form.

You study hard or you will do badly in this course.

Solution.

Let $\sim p$: You study hard. (So p: You do not study hard.)

q: You will do badly in this course.

The given statement is $\sim p \vee q$. This is equivalent to $p \to q$, or

If you do not study hard, then you will do badly in this course.

Negation of a Conditional Statement

We can use the expression of $p \to q$ as an "or" statement to derive its negation.

$$\begin{aligned} \sim (p \to q) &\equiv \sim (\sim p \vee q) \\ &\equiv \sim (\sim p) \wedge \sim q \quad \text{(De Morgan's laws)} \\ &\equiv p \wedge \sim q \quad \text{(Double negation)} \end{aligned}$$

That is,

$$\boxed{\sim (p \to q) \equiv p \wedge \sim q}$$

Example

The negation of *"If it is sunny, then I go to the park."* is

It is sunny and I do not go to the park.

The Contrapositive of a Conditional Statement

Definition

The **contrapositive** of a conditional statement of the form *"If p, then q"* is the statement

"If $\sim q$, then $\sim p$."

That is, the contrapositive of $p \to q$ is $\sim q \to \sim p$.

Example

The contrapositive of the statement

"If you have worked hard in this course, then you will pass."

is the statement

"If you do not pass, then you have not worked hard in this course."

The Contrapositive

Fact

A conditional statement and its contrapositive are logically equivalent. That is,

$$\boxed{p \to q \equiv \sim q \to \sim p}$$

Converse and Inverse

Definition

1. The **converse** of a conditional statement $p \to q$ is the statement $q \to p$.
2. The **inverse** of a conditional statement $p \to q$ is the statement $\sim p \to \sim q$.

Example

Consider the statement

"If it is snowing, then it is cold outside."

Its converse is the statement

"If it is cold outside, then it is snowing."

Its inverse is the statement

"If it is not snowing, then it is not cold outside."

Converse and Inverse

Remark

1. **Caution:** A conditional statement and its converse are **not** logically equivalent, i.e.

$$p \to q \not\equiv q \to p$$

   For example, the statement *"If it is snowing, then it is cold outside"* is not equivalent to the statement *"If it is cold outside, then it is snowing"*.

2. A conditional statement and its inverse are **not** logically equivalent, i.e.

$$p \to q \not\equiv \sim p \to \sim q$$

3. The converse and inverse are, however, logically equivalent, i.e.

$$q \to p \equiv \sim p \to \sim q$$

Other ways of expressing the conditional statement $p \to q$

- **p only if q**

  The statement *"p only if q"* means *"if not q, then not p"*, or equivalently $p \to q$

- **p is a sufficient condition for q**

  The statement *"p is a sufficient condition for q"* (or *"p is sufficient for q"*) means that the occurrence of p is sufficient to guarantee the occurrence of q, i.e. if p occurs, then q also occurs.

- **q is a necessary condition for p**

  The statement *"q is a necessary condition for p"* (or *"q is necessary for p"* means that if q does not occur, then neither will p, i.e. $\sim q \to \sim p$, which is equivalent to $p \to q$.

Other ways of expressing the conditional statement $p \rightarrow q$

- **q if p**
- **q whenever p**
- **p implies q**

Other ways of expressing $p \to q$

Example

Express the following statements in the form *"If p, then q"*:

1. The Leafs will win the Stanley Cup only if they have made the playoffs.

   If the Leafs win the Stanley Cup, then they have made the playoffs.

2. Being continuous is a necessary condition for the function f to be differentiable.

   If the function f is differentiable, then it is continuous.

3. John's reaching 18 years of age and being a Canadian citizen are sufficient conditions for him to vote in the next election.

   If John has reached 18 years of age and is a Canadian citizen, then he can vote in the next election.

The biconditional

Definition

Let p and q be statement variables. The **biconditional of** p **and** q is the statement *"p if and only if q"*, denoted by $\boldsymbol{p \leftrightarrow q}$ (or p iff q). It is true if both p and q have the same truth values, and false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example

Let n be a given integer. Let p be the statement *"n is even"* and q the statement *"n is divisible by 2"*. The statement $p \leftrightarrow q$ is

"n is even if and only if n is divisible by 2

The biconditional

It can be shown (exercise) that

$$\boxed{p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)}$$

That is, *"p if and only if q"* is equivalent to *"p only if q and p if q"*.

Remark

The statement *"p if and only if q"* is sometimes expressed as

"p is a necessary and sufficient condition for q"

Example

Express as an if-and-only-if statement:

"Necessary and sufficient conditions for a matrix A to be invertible are that A is square and $\det(A) \neq 0$."

Solution: A matrix A is invertible if and only if A is square and $\det(A) \neq 0$.

Remarks on conditional statements

- In logic, a hypothesis and conclusion need not have related subject matters.

  e.g. *"If 24 is divisible by 6, then the Pope is Catholic."*

- In informal language, conditionals are sometimes used to mean biconditionals.

  e.g. *"If you eat your vegetables, then you can have dessert."*
  vs. *"You can have dessert if and only if you eat your vegetables."*

  In formal logic, these two are different!

2.3: Valid and Invalid Arguments

Arguments

The central notion of deductive logic is that of an **argument**, a sequence of statements leading to a conclusion.

Consider the following classical example:

	If Socrates is a man, then Socrates is mortal.
	Socrates is a man.
$\therefore$	Socrates is mortal.

If the first two statements are true, then we can deduce the truth of the third.

We can write this argument in an abstract form:

$$\begin{array}{ll} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

Arguments

Definition

An **argument** is a sequence of statements. All statements except the final one are called **premises** (or **assumptions** or **hypotheses**), and the final statement is called the **conclusion**. (We normally place the symbol $\therefore$ (meaning therefore) before the conclusion.)

$$\begin{array}{rc} & P_1 \\ & P_2 \\ & \vdots \\ & P_n \\ \hline \therefore & C \end{array}$$

Validity of Arguments

Definition

An argument is said to be **valid** if any truth values of the statement variables involved which make the premises all true, also make the conclusion true.

Method for testing validity of an argument

1. Identify the premises and conclusion of the argument.
2. Construct a truth table showing truth values for all premises and the conclusion.
3. Identify all rows in which all premises are true. Such a row is called a **critical row**.
4. If the conclusion is true in each critical row, then the argument is **valid**.
5. If there is at least one critical row in which the conclusion is false, then the argument is **invalid**.

Validity of Arguments

Example

Is the following argument valid?

$$\begin{array}{rl} & (p \vee \sim q) \rightarrow r \\ & \sim q \rightarrow (r \wedge \sim p) \\ \hline \therefore & \sim p \vee q \end{array}$$

Construct a truth table.

p	q	r	$\sim p$	$\sim q$	$p \vee \sim q$	$r \wedge \sim p$	$(p \vee \sim q) \rightarrow r$	$\sim q \rightarrow (r \wedge \sim p)$	$\sim p \vee q$
T	T	T	F	F	T	F	**T**	**T**	**T**
T	T	F	F	F	T	F	F	T	T
T	F	T	F	T	T	F	T	F	F
T	F	F	F	T	T	F	F	F	F
F	T	T	T	F	F	T	**T**	**T**	**T**
F	T	F	T	F	F	F	**T**	**T**	**T**
F	F	T	T	T	T	T	**T**	**T**	**T**
F	F	F	T	T	T	F	F	F	T

In every row where the premises are all true, the conclusion is also true. So the argument is valid!