

## Example: A truth table with three variables

Construct a truth table for the statement

$$\sim p \vee (q \wedge (p \vee \sim r))$$

$p$	$q$	$r$	$\sim p$	$\sim r$	$p \vee \sim r$	$q \wedge (p \vee \sim r)$	$\sim p \vee (q \wedge (p \vee \sim r))$
$T$	$T$	$T$	$F$	$F$	$T$	$T$	$T$
$T$	$T$	$F$	$F$	$T$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	$F$	$T$	$F$	$F$
$T$	$F$	$F$	$F$	$T$	$T$	$F$	$F$
$F$	$T$	$T$	$T$	$F$	$F$	$F$	$T$
$F$	$T$	$F$	$T$	$T$	$T$	$T$	$T$
$F$	$F$	$T$	$T$	$F$	$F$	$F$	$T$
$F$	$F$	$F$	$T$	$T$	$T$	$F$	$T$

# Logical Equivalence

Consider the statements “I am a mathematician and I live in Toronto” and “I live in Toronto and I am a mathematician”. Clearly these are saying the same thing. Let us formalize what it means for two statements to be equivalent.

## Definition

Two statement forms  $P$  and  $Q$  are **logically equivalent** if they have the same truth values for all possible truth values of the variables ( $p, q, r$ , etc.) involved. If  $P$  and  $Q$  are equivalent, we write  $P \equiv Q$ .

## Determining logical equivalence

Method for checking if  $P$  and  $Q$  are logically equivalent.

1. Construct a truth table which has columns for  $P$  and  $Q$ .
2. If the truth values of  $P$  and  $Q$  agree in every row, then  $P$  and  $Q$  are equivalent. Otherwise, if there is a row in which the truth values for  $P$  and  $Q$  differ, then the statement forms are not equivalent.

### Example

Show that  $p \equiv \sim(\sim p)$ .

$p$	$\sim p$	$\sim(\sim p)$
$T$	$F$	$T$
$F$	$T$	$F$

The columns for  $p$  and  $\sim(\sim p)$  are identical, so  $p \equiv \sim(\sim p)$ .

# Logical Equivalence

## Example

Show that the statements  $\sim (p \vee q)$  and  $\sim p \vee \sim q$  are **not** logically equivalent (i.e.  $\sim (p \vee q) \not\equiv \sim p \vee \sim q$ ).

$p$	$q$	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$		$\sim p \vee \sim q$
$T$	$T$	$F$	$F$	$T$	$F$		$F$
$T$	$F$	$F$	$T$	$T$	$F$	$\neq$	$T$
$F$	$T$	$T$	$F$	$T$	$F$	$\neq$	$T$
$F$	$F$	$T$	$T$	$F$	$T$		$T$

The statements do not always have the same truth values, so they are not equivalent.

# DeMorgan's Laws

## Example

Show that  $\sim (p \vee q) \equiv \sim p \wedge \sim q$ .

$p$	$q$	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
$T$	$T$	$F$	$F$	$T$	$F$	$F$
$T$	$F$	$F$	$T$	$T$	$F$	$F$
$F$	$T$	$T$	$F$	$T$	$F$	$F$
$F$	$F$	$T$	$T$	$F$	$T$	$T$

Similarly, it can be shown that  $\sim (p \wedge q) \equiv \sim p \vee \sim q$  (done in textbook).

These equivalences are known as De Morgan's Laws:

$$\begin{array}{l} \sim (p \vee q) \equiv \sim p \wedge \sim q \\ \sim (p \wedge q) \equiv \sim p \vee \sim q \end{array}$$

# De Morgan's Laws

## Example

Use De Morgan's laws to write the negation of the inequality  $0 \leq x < 100$ .

**Solution:** The inequality  $0 \leq x < 100$  is equivalent to

$$0 \leq x \text{ and } x < 100,$$

so its negation is

$$0 \not\leq x \text{ or } x \not< 100,$$

or equivalently,

$$x < 0 \text{ or } x \geq 100.$$

# Tautologies and Contradictions

## Definition

A **tautology** is a statement which is always true, no matter what truth values are assigned to its statement variables.

A **contradiction** is a statement which is always false, no matter what truth values are assigned to its statement variables.

# Tautologies and Contradictions

## Example

Use truth tables to show that  $p \vee \sim p$  is always a tautology, and  $p \wedge \sim p$  is a contradiction.

$p \vee \sim p$ :

$p$	$\sim p$	$p \vee \sim p$
$T$	$F$	$T$
$F$	$T$	$T$

Every entry in the column for  $p \vee \sim p$  is T: tautology.

$p \wedge \sim p$ :

$p$	$\sim p$	$p \wedge \sim p$
$T$	$F$	$F$
$F$	$T$	$F$

Every entry in the column for  $p \wedge \sim p$  is F: contradiction.

# Tautologies and Contradictions

## Notation

We will use the symbol **t** to denote a tautology and **c** to denote a contradiction.

## Example

Show that  $p \vee \mathbf{t} \equiv \mathbf{t}$ .

$p$	<b>t</b>	$p \vee \mathbf{t}$
$T$	$T$	$T$
$F$	$T$	$T$

The columns for  $p \vee \mathbf{t}$  and **t** are identical, so  $p \vee \mathbf{t} \equiv \mathbf{t}$ .

# Useful Logical Equivalences

## 1. Commutative laws:

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

## 2. Associative laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

## 3. Distributive laws:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

## 4. Identity laws:

$$p \wedge \mathbf{t} \equiv p$$

$$p \vee \mathbf{c} \equiv p$$

# Useful Logical Equivalences

5. Negation laws:

$$p \vee \sim p \equiv \mathbf{t}$$

$$p \wedge \sim p \equiv \mathbf{c}$$

6. Double negative law:

$$\sim(\sim p) \equiv p$$

7. Idempotent laws:

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

8. Universal bound laws:

$$p \vee \mathbf{t} \equiv \mathbf{t}$$

$$p \wedge \mathbf{c} \equiv \mathbf{c}$$

# Useful Logical Equivalences

9. De Morgan's laws:

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

10. Absorption laws:

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

11. Negations of **t** and **c**:

$$\sim \mathbf{t} \equiv \mathbf{c}$$

$$\sim \mathbf{c} \equiv \mathbf{t}$$

(We have already verified some of these rules. The others may be done similarly.)

# Using Logical Equivalences to Simplify Statement Forms

## Example

Show that  $\sim (\sim p \vee q) \vee (p \wedge q) \equiv p$ .

## Solution.

$$\begin{aligned} & \sim (\sim p \vee q) \vee (p \wedge q) \\ \equiv & (\sim (\sim p) \wedge \sim q) \vee (p \wedge q) && \text{(De Morgan)} \\ \equiv & (p \wedge \sim q) \vee (p \wedge q) && \text{(Double negation)} \\ \equiv & p \wedge (\sim q \vee q) && \text{(Distributive law)} \\ \equiv & p \wedge (q \vee \sim q) && \text{(Commutative law)} \\ \equiv & p \wedge \mathbf{t} && \text{(Negation law)} \\ \equiv & p && \text{(Identity law)} \end{aligned}$$

## 2.2: Conditional Statements

# Conditional Statements

In this section, we will study **conditional statements**, i.e. those of the form

If  $p$ , then  $q$ .

## Example

- ▶ If  $n$  is odd, then  $n$  is not a multiple of 2.

## Notation

The statement “If  $p$ , then  $q$ ” is written as

$$p \rightarrow q$$

Here,  $p$  is called the **hypothesis** (or premise) and  $q$  is called the **conclusion**.

# Conditional Statements

- ▶ When is the statement  $p \rightarrow q$  true?
- ▶ If the hypothesis  $p$  is true, then the conclusion  $q$  should also be true in order to make  $p \rightarrow q$  true.
- ▶ If the hypothesis  $p$  is false, then no promise is made as to the truth or falsity of the conclusion  $q$ . By convention,  $p \rightarrow q$  is true whenever  $p$  is false.
- ▶ The **only** instance in which  $p \rightarrow q$  is false is if  $p$  is true and  $q$  is false.