Chapter 2: The Logic of Compound Statements

MTH 110, Fall 2011

2.1: Logical Form and Logical Equivalence

Introduction

- Deductive reasoning forms much of the basis of mathematical theory.
- In mathematics, we use deductive logic to construct arguments and prove the truth of statements.
- In computer science, logic provides a framework to prove that algorithms work.
- Symbolic logic has also provided a theoretical basis for areas of computer science such as digital logic circuit design, relational database theory, automata theory and computability, and artificial intelligence.

Introduction: Arguments

- The central concept of deductive logic is the argument form.
- An *argument* is a sequence of statements whose goal is to show that a particular assertion is true.
- The assertion at the end is called the *conclusion*, and the preceding statements are called *premises*.
- The purpose of an argument is to show that the truth of the conclusion follows from the truth of the premises.

Example

If x is a real number such that $x < -3$ or $x > 3$, then $x^2 > 9$. Therefore, if $x^2 \not> 9$, then $x \not< -3$ and $x \not> 3$.

We can use variables (e.g. p, q, r) to represent statements that occur in the argument. This example has the following form:

> If p or q, then r. Therefore, if not r, then not p and not q.

Example

Fill in the blanks in argument (b) so that it has the same form as argument (a). Then write the common form of the argument using letters to replace the individual statements.

(a) If the train is late, then I will miss my meeting.
The train is late.
Therefore, I will miss my meeting.

(b) If ___(1)___, then I will get an A.
MTH 110 is fun.
Therefore, ___(2)___.

Solution.

(1) MTH 110 is fun.
(2) I will get an A.

The common form of the arguments:

If p, then q.
p
Therefore, q.

Statements

Definition

A **statement** (or **proposition**) is a sentence which is either true or false, but not both.

Example (Statements)

1. *"The area of a square of side n is n^2."* (A true statement)
2. *"64 is divisible by 3."* (A statement which is false)

Example (Non-statements)

1. *"Is it sunny today?"*
   Not a statement, since a question can neither be true nor false.
2. $x + y > 3$
   Whether this sentence is true depends on the values of x and y, so it is not a statement as is. **However**, if we agree on particular values of x and y, then it becomes a statement.

Compound statements

- We frequently use variables to represent statements. (E.g. p)
- Starting with simple statements, we can build more complex statements using **connectives**.
- Some important connectives:
  1. $\sim$ (not)
  2. $\wedge$ (and)
  3. $\vee$ (or)

Compound statements

A statement formed from statement variables with connectives is called a **compound statement**.

- Let p and q be statements. Then
  1. $\sim p$, read "not p", is the **negation of p**.
  2. $p \wedge q$, read "p and q", is the **conjunction of p and q**.
  3. $p \vee q$, read "p or q", is the **disjunction of p and q**.

Example

$\sim p \wedge (q \vee \sim r)$ is a compound statement.

Remark

There is an order of operations for statement connectives.
Evaluate expressions in brackets first, and then $\sim$ before $\wedge$ or $\vee$.
So, for example, $\sim p \vee q$ is the same as $(\sim p) \vee q$ (either not p or else q), but **not** $\sim (p \vee q)$ (neither p nor q).

Compound statements

Example

Let p: It is cold.
q: It is cloudy.
r: It is snowing.

- It is cloudy or it is snowing.

$$q \vee r$$

- It is cold and it is not snowing.

$$p \wedge \sim r$$

- It is cold and cloudy and snowing.

$$p \wedge q \wedge r$$

Compound statements

Some common English phrases which can be translated using $\sim$, $\wedge$, $\vee$:

- p but q: Means "p and q".
- neither p nor q: Means "not p and not q".

Example

Let p: John wins the lottery.
q: John quits his job.

- The statement "John wins the lottery but does not quit his job" can be translated as

$$p \wedge \sim q$$

- The statement "John neither wins the lottery nor quits his job" can be translated as

$$\sim p \wedge \sim q$$

Statements involving inequalities

Mathematical inequalities are sometimes *and* or *or* statements written in a compressed form.
For instance,

1. $x \leq a$ means $(x < a)$ or $x = a$
2. $a < x < b$ means $a < x$ and $x < b$.

Example

Let p: "$1 < x$, q: "$x < 2$", r: $x = 2$. Express symbolically:

1. $x \leq 2$

$$q \vee r$$

2. $x \geq 2$

$$\sim q$$

3. $1 < x \leq 2$

$$p \wedge (q \vee r)$$

Truth values

- Recall that a statement must be either true or false, but not both. That is, a statement has a **truth value** of true (T) or false (F).
- For compound statements, the truth value depends on the truth values of the variables involved.
- We determine truth or falsity by means of a **truth table**.
- A truth table contains a row for each combination of truth values of the variables. Columns represent statements. Usually, the rightmost column contains truth values for the whole statement.
- If you have n statement variables, then you will need 2^n rows.

Truth table for negation

Definition

Let p be a statement. The **negation of p**, denoted $\sim p$, is false when p is true and true when p is false.

Truth table:

p	$\sim p$
T	F
F	T

Truth table for conjunction

Definition

Let p and q be statements. The **conjunction of $\boldsymbol{p}$ and $\boldsymbol{q}$**, denoted $\boldsymbol{p \wedge q}$, is the statement which is true if both p and q are true, and false otherwise.

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for disjunction

Definition

Let p and q be statements. The **disjunction of p and q**, denoted $\boldsymbol{p \vee q}$, is the statement which is true if one of p and q is true (or both are true), and false if both p and q are false.

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

In common language, the word "or" can be used in an inclusive or exclusive sense.

Example

Inclusive: "You will pass if you have an overall average of 50% or you get 75% on the final."

Exclusive: "The Leafs or the Canadiens will win the Stanley Cup this year."

In logic, we must be clear of the distinction. The symbol $\vee$ denotes the inclusive sense of or.

Exclusive or

Consider the statement $(p \vee q) \wedge \sim (p \wedge q)$. When is it true?

Construct a truth table.

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \wedge q) \vee (\sim p \wedge \sim q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

This statement is true when exactly one of p and q is true. It is called the **exclusive or** of p and q, denoted $\boldsymbol{p \oplus q}$ (or sometimes $p \text{ XOR } q$).