MAT 2384 Assignment #3 : Solutions

1. $y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = 1$

the characteristic equation is $\lambda^2 + \lambda - 2 = 0$ or $(\lambda+2)(\lambda-1) = 0$
so $\lambda_1 = -2$, $\lambda_2 = 1$ (Case I) and the general solution
is $y(x) = c_1 e^{-2x} + c_2 e^{x}$

$y(0) = 4 \Rightarrow 4 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 4$
$y'(x) = -2c_1 e^{-2x} + c_2 e^{x}$
$y'(0) = 1 \Rightarrow 1 = -2c_1 e^0 + c_2 e^0 \Rightarrow -2c_1 + c_2 = 1$

$\Big\} \quad c_1 = 1, \quad c_2 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = e^{-2x} + 3e^{x}}$

2. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$

the char. eq. is $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$ (Case III)
the general solution is $y(x) = c_1 e^{3x} + c_2 x e^{3x}$

$y(0) = 0 \Rightarrow 0 = c_1 e^0 + c_2 (0) e^0 \Rightarrow c_1 = 0$
$y'(x) = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x}$
$y'(0) = 2 \Rightarrow 2 = 3c_1 e^0 + c_2 e^0 + 3c_2 (0) e^0 \Rightarrow 3c_1 + c_2 = 2 \Rightarrow c_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2x e^{3x}}$

3. $y'' - 3y' - 10y = 0, \quad y(0) = 2, \quad y'(0) = 3$

char. eq. is $\lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0 \Rightarrow \lambda_1 = -2$
and $\lambda_2 = 5$ (Case I) and the general solution is
$y(x) = c_1 e^{-2x} + c_2 e^{5x}$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 2$
$y'(x) = -2c_1 e^{-2x} + 5c_2 e^{5x}$
so $y'(0) = 3 \Rightarrow 3 = -2c_1 e^0 + 5c_2 e^0 \Rightarrow -2c_1 + 5c_2 = 3$

$\Big\} \quad c_1 = c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = e^{-2x} + e^{5x}}$

4. $y'' + 2y = 0$, $y(0) = \pi$, $y'(0) = 2$

char. eq. is $\lambda^2 + 2 = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{2}\,i$ (case II)
the general solution is $y(x) = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$
$y(0) = \pi \Rightarrow \pi = c_1 \cos(0) + c_2 \sin(0) \Rightarrow c_1 = \pi$
$y'(x) = -\sqrt{2}c_1 \sin(\sqrt{2}x) + \sqrt{2}c_2 \cos(\sqrt{2}x)$
$y'(0) = 2 \Rightarrow 2 = -\sqrt{2}c_1 \sin(0) + \sqrt{2}c_2 \cos(0) \Rightarrow c_2 = \sqrt{2}$

$\therefore$ the unique solution is $\boxed{y(x) = \pi\cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)}$

5. $y'' + 2\sqrt{3}y' + 3y = 0$, $y(0) = \sqrt{3}$, $y'(0) = 0$

the char. eq. is $\lambda^2 + 2\sqrt{3}\lambda + 3 = (\lambda + \sqrt{3})^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -\sqrt{3}$ (case III)
and the general solution is $y(x) = c_1 e^{-\sqrt{3}x} + c_2 x e^{-\sqrt{3}x}$
$y(0) = \sqrt{3} \Rightarrow \sqrt{3} = c_1 e^0 + c_2(0)e^0 \Rightarrow c_1 = \sqrt{3}$
$y'(x) = -\sqrt{3}c_1 e^{-\sqrt{3}x} + c_2 e^{-\sqrt{3}x} - \sqrt{3}c_2 x e^{-\sqrt{3}x}$
$y'(0) = 0 \Rightarrow 0 = -\sqrt{3}c_1 e^0 + c_2 e^0 - \sqrt{3}c_2(0)e^0 \Rightarrow c_2 = \sqrt{3}c_1 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = \sqrt{3}e^{-\sqrt{3}x} + 3xe^{-\sqrt{3}x}}$

6. $y'' - 6y' + 13y = 0$, $y(0) = 2$, $y'(0) = 0$

so the char. eq. is $\lambda^2 - 6\lambda + 13 = 0$ and the roots are $\lambda_{1,2} = \dfrac{6 \pm \sqrt{(-6)^2 - 4(13)}}{2} = 3 \pm 2i$ (case II)

and the general solution is $y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x)$
$y(0) = 2 \Rightarrow 2 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) \Rightarrow c_1 = 2$

$$y'(x) = 3c_1 e^{3x}\cos(2x) - 2c_1 e^{3x}\sin(2x) + 3c_2 e^{3x}\sin(2x) + 2c_2 e^{3x}\cos(2x)$$

$y'(0) = 0 \Rightarrow 0 = 3c_1 e^0\cos(0) - 2c_1 e^0 \sin(0) + 3c_2 e^0 \sin(0) + 2c_2 e^0\cos(0)$

$\Rightarrow 3c_1 + 2c_2 = 0 \Rightarrow c_2 = -3$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{3x}\cos(2x) - 3e^{3x}\sin(2x)}$

7. $x^2y'' - 4xy' + 6y = 0, \quad y(1) = 4, \quad y'(1) = 10$

the char. eq. is $m(m-1) - 4m + 6 = m^2 - 5m + 6 = 0$

or $(m-2)(m-3) = 0 \Rightarrow m_1 = 2, m_2 = 3$ (case I)

and the general solution is $y(x) = c_1x^2 + c_2x^3$

$y(1) = 4 \Rightarrow 4 = c_1(1)^2 + c_2(1)^3 \Rightarrow c_1 + c_2 = 4$

$y'(x) = 2c_1x + 3c_2x^2$

so $y'(1) = 10 \Rightarrow 10 = 2c_1(1) + 3c_2(1)^2 \Rightarrow 2c_1 + 3c_2 = 10$

$c_1 = 2$
$c_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2x^2 + 2x^3}$

8. $x^2y'' + 5xy' + 4y = 0, \quad x > 0, \quad y(1) = 1/2, \quad y'(1) = -2$

char. eq. is $m(m-1) + 5m + 4 = m^2 + 4m + 4 = (m+2)^2 = 0$

so $m_1 = m_2 = -2$ (case III) and the general solution is

$y(x) = c_1x^{-2} + c_2x^{-2}\ln x$

$y(1) = 1/2 \Rightarrow 1/2 = c_1(1)^{-2} + c_2(1)^{-2}\ln(1) \Rightarrow c_1 = 1/2$

$y'(x) = -2c_1x^{-3} - 2c_2x^{-3}\ln x + c_2x^{-3}$

thus $y'(1) = -2 \Rightarrow -2 = -2c_1(1)^{-3} - 2c_2(1)^{-3}\ln(1) + c_2(1)^{-3}$

which gives $-2c_1 + c_2 = -2 \Rightarrow c_2 = -1$

$\therefore$ the unique solution is $\boxed{y(x) = \frac{1}{2}x^{-2} - x^{-2}\ln x}$

9. $x^2y'' - 5xy' + 13y = 0, \quad x>0, \quad y(1)=0, \quad y'(1)=5$

char. eq. is $m(m-1) - 5m + 13 = m^2 - 6m + 13 = 0$
and so $m_{1,2} = 3 \pm 2i$ (Case II)
and the general solution is $y(x) = c_1 x^3\cos(2\ln x) + c_2 x^3 \sin(2\ln x)$
$y(1) = 0 \Rightarrow 0 = c_1(1)^3\cos(0) + c_2(1)^3\sin(0) \Rightarrow c_1 = 0$
$y'(x) = 3c_1x^2\cos(2\ln x) - 2c_1x^2\sin(2\ln x) + 3c_2x^2\sin(2\ln x)$
$+ 2c_2x^2\cos(2\ln x)$
so $y'(1)=5 \Rightarrow 5 = 3c_1(1)^2\cos(0) - 2c_1(1)^2\sin(0) + 3c_2(1)^2\sin(0)$
$+ 2c_2(1)^2\cos(0)$
or $3c_1 + 2c_2 = 5 \Rightarrow c_2 = 5/2$

$\therefore$ the unique solution is $\boxed{y(x) = \frac{5}{2}x^3\sin(2\ln x)}$