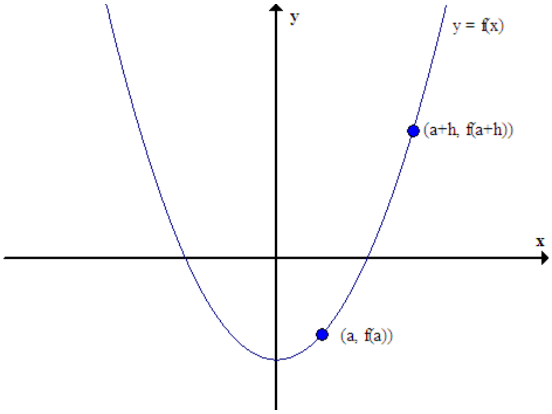
Chapter 1

Goals

- to understand average and instantaneous rates of change and their connections to the slopes of secant and tangent lines
- to be able to calculate average rates of change and estimate instantaneous rates of change from an equation, a graph or a data table
- to understand the concept of a limit and be able to use the basic limit properties to find limits of sequences and functions
- to understand the concept of continuity and be able to tell if a function is continuous or not at a point
- to be able to recognize the common discontinuities that can occur on the graph of a function
- to be able to recognize the indeterminate form $\frac{0}{0}$ when it occurs in the evaluation of a limit and know what to do to find the limit
- to understand the definition of the derivative of a function at a point and as a function and be able to use the definitions to calculate them
- to be comfortable with the different notations for the derivative

Rates of Change and the Slopes of Curves

Suppose we have a function $y = f(x)$. If we change x from $x = a$ to $x = a + \Delta x = a + h$, for a step or difference in x of $\Delta x = h$, then the value of the function (asuming it's not constant), will change from $f(a)$ to $f(a + h)$.

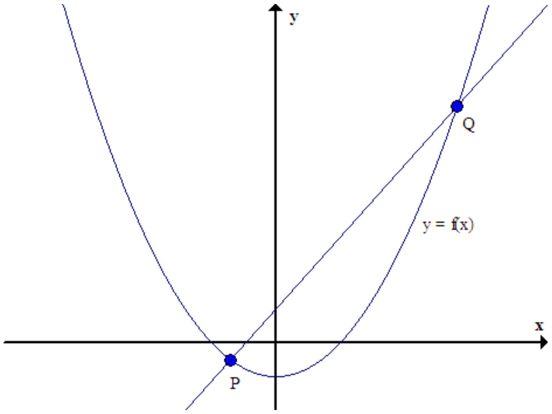


If we look at the change in the value of the function, $\Delta f = f(a + h) - f(a)$, relative to the change in the independent variable x, $\Delta x = (a + h) - a = h$, we'll have $\dfrac{\Delta f}{\Delta x} = \dfrac{f(a + h) - f(a)}{h}$, which is the average rate of change of the function on the interval $a \leq x \leq a + h$.

The most familiar example of this is probably the velocity of a moving object. if you drive a distance of 120 km in an hour and a half, your average speed for the trip is $\dfrac{120 \text{ km}}{1.5 \text{ h}} = 80$ km/h (and this is a change in position divided by a change in time).

Let's go back to what we had above. Notice that the expression $\dfrac{\Delta f}{\Delta x} = \dfrac{f(a + h) - f(a)}{h}$ (called the difference quotient) would represent the slope of a straight line through the two points $(a, f(a))$ and $(a + h, f(a + h))$ on the curve.

A line that passes through two points P and Q on a curve $y = f(x)$ is called a secant line.



Example:
Find the slope of the secant line passing through the points $(-1, 1)$ and $(2, 4)$ on the curve $y = x^2$.

$\dfrac{\Delta y}{\Delta x} = \dfrac{4-1}{2-(-1)} = \dfrac{3}{3} = 1$ and the line is $y = x + 2$ (*can you see that?*).

Example:
Suppose the population of a small town was measured every year from 2001 to 2010.

Year	Population
2001	6210
2002	6347
2003	6523
2004	6704
2005	6851
2006	6975
2007	7087
2008	7214
2009	7326
2010	7472

(i) What was the average rate of change of the population over the entire period?
(ii) How about over 2006 to 2010?

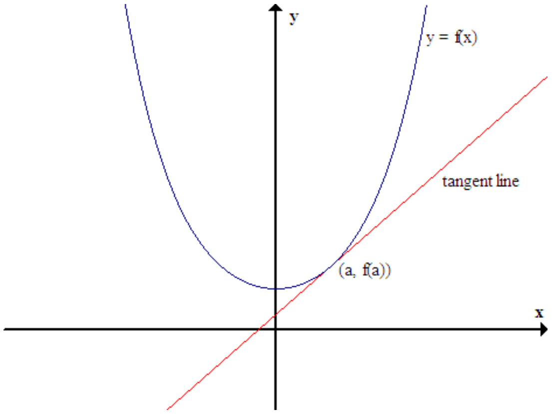
(i) $\dfrac{\Delta P}{\Delta t} = \dfrac{7472 - 6210}{2010 - 2001} \approx 140$ people/year

(ii) $\dfrac{\Delta P}{\Delta t} = \dfrac{7472 - 6975}{2010 - 2006} \approx 124$ people/year

In our driving example above, we had that the average speed over the trip was 80 km/h.

It is unlikely that we were actually driving at exactly 80 km/h the entire 1.5 hours – more likely, at times we were going faster and at other times, slower. Also, the average does not tell us what our speed was at any particular time, which would be an instantaneous velocity (speed at one particular instant).

How could we find the instantaneous rate of change of a function $y = f(x)$ at the value $x = a$? By recognizing that this would correspond to the "slope" of the curve at this point, which would have to be equal to the slope of the tangent line to the curve at the point.



The tangent line to a curve at a point $P = (a, f(a))$ is the straight line that passes through the point and best approximates the curve near the point.

So if we have the graph of the function, we could draw the tangent line and find its slope. (*But this would only be an approximation or estimate as we cannot draw the tangent line with perfect accuracy.*)

If we have a table of values, we could calculate the average rate of change over as short an interval as possible that contains the point of interest (*but again, only an estimate*).

Example:
The instantaneous rate of change m of the population of the town in 2004 is approximately

$$m \approx \frac{\Delta P}{\Delta t} = \frac{P(2005) - P(2004)}{2005 - 2004} = \frac{6851 - 6704}{1} = 147 \text{ people/year.}$$

Notice that either way (graphically or numerically), the best we can do is an estimate. Why do we have this problem? Because we only know that the tangent line passes through the point $(a, f(a))$ and nothing else. We cannot calculate the slope (or find the equation) of the line knowing only a single point. So, clearly, if we wish to calculate instantaneous rates of change (*which we do*), we need to figure out a way to do so.

Rates of Change Using Equations

If we have the equation of a function, $y = f(x)$, we can make more accurate estimates of the instantaneous rate of change when $x = a$ by finding the average rate of change over a small interval $a \leq x \leq a + h$.

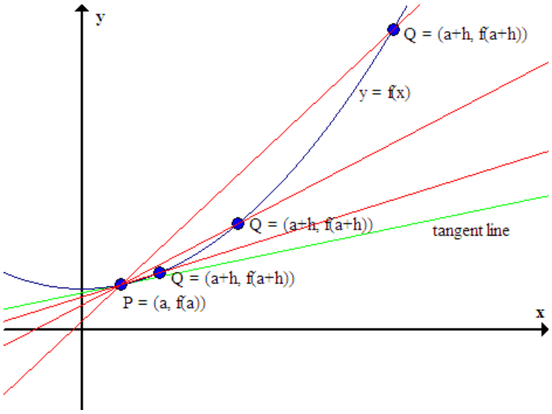
Example:
The position (in meters) of a moving object is given by $s(t) = 2t^2 + 3t + 2$, where t is measured in seconds. What is the instantaneous velocity of the object at time $t = 2$ s ?

The slope of the secant line (which we are using as an approximation of the tangent line) passing through $P = (2, s(2))$ and $Q = (2 + h, s(2 + h))$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(2+h) - s(2)}{(2+h) - 2}$$
$$= \frac{2(2+h)^2 + 3(2+h) + 2 - (2(2)^2 + 3(2) + 2))}{h}$$
$$= \frac{2(4 + 4h + h^2) + 6 + 3h + 2 - 16}{h}$$
$$= \frac{8 + 8h + 2h^2 + 6 + 3h + 2 - 16}{h}$$
$$= \frac{11h + 2h^2}{h}$$
$$= 11 + 2h$$

so if $h = 1$ s, $\dfrac{\Delta s}{\Delta t} = 13$ m/s
so if $h = 0.1$ s, $\dfrac{\Delta s}{\Delta t} = 11.2$ m/s
so if $h = 0.01$ s, $\dfrac{\Delta s}{\Delta t} = 11.02$ m/s
and so it appears that the instantaneous velocity of the object at time $t = 2$ s is 11 m/s.

What are we actually doing here? We're using secant lines to approximate the tangent line.



The smaller h is, the closer Q is to P and the better the secant line approximates the tangent line. And as we shrink h to zero, the difference quotient $\dfrac{f(a+h)-f(a)}{h}$, which is the slope of the secant line or the average rate of change on the interval $a \leq x \leq a+h$ becomes closer and closer to the slope of the tangent line or the instantaneous rate of change at $x = a$.

But what do we mean by shrinking h to 0 and something becoming closer and closer to something else?

Limits

The set of natural numbers is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. An infinite sequence is an infinite list of numbers generated by a function $f(n) = a_n$ whose domain is $\mathbb{N}$.

Example:

$2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \ldots$

The general term here is $a_n = \dfrac{2}{3^n}$ and as n gets bigger, the values will get smaller and smaller or closer to 0. There is an a_n for all n, no matter how large n is and we can denote the idea of n getting larger and larger without bound by saying that n approaches infinity, which we write as $n \to \infty$. As $n \to \infty$, $a_n \to 0$. We can write this as a limit: $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3^n} = 0$.

Example:

If $a_n = (-1)^n$, we have the sequence

$1, -1, 1, -1, \ldots$

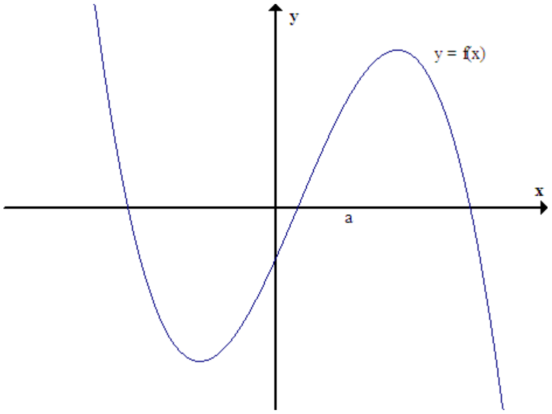
Here, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$ does not exist because the terms in the sequence are not approaching a single value.

Example:
$1, 2, 4, 8, 16, 32, \ldots$
Here $a_n = 2^n$ and then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^n = \infty$. This limit does not exist as the terms in the sequence are growing without bound, getting larger and larger, and so they do not approach a single (finite) value.

∞ is not a number – it represents the idea of unbounded growth.

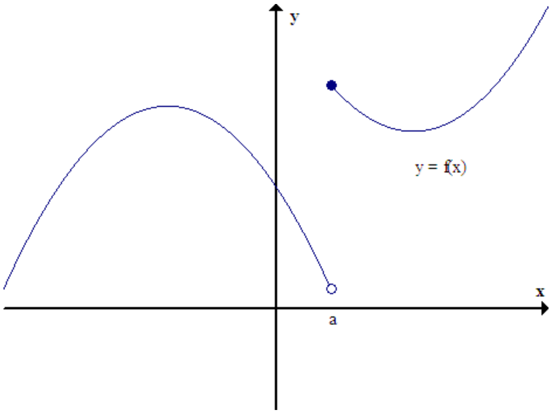
If $\lim\limits_{n\to\infty} a_n = L$, where L is a unique and finite number, then the sequence $\{a_n\}$ has a limit as $n \to \infty$ and is said to <u>converge to L</u> which means that as n gets larger and larger, the values of a_n approach L.

Given a function $f(x)$, we can also look at what happens as $x \to a$, *ie* take $\lim\limits_{x\to a} f(x)$.



We can approach a from either the left side, where $x < a$ or the right side, where $x > a$. This leads to <u>the limit from the left</u> $\lim\limits_{x\to a^-} f(x)$ and <u>the limit from the right</u> $\lim\limits_{x\to a^+} f(x)$.

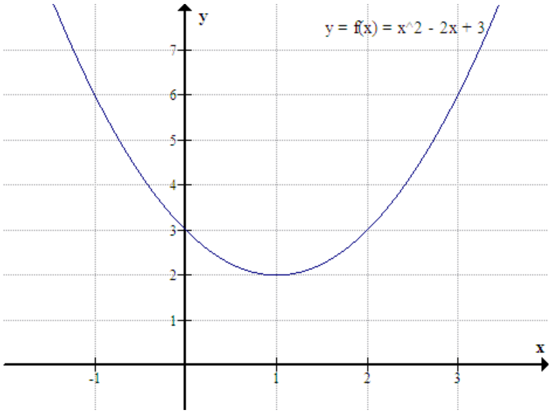
Suppose that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but are different (not equal).



Then $\lim_{x\to a} f(x)$ cannot exist as $f(x)$ is not approaching a single value as $x \to a$.

What if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$? Then $\lim_{x\to a} f(x)$ must exist and be the same value.

Example:
Consider the function $f(x) = x^2 - 2x + 3$. What is $\lim_{x\to 2} f(x)$?



We can see from the graph that as $x \to 2^-$ (*ie* from the left, $x = 1.9, 1.99, 1.999, \ldots$), the value of the function will increase up to 3. And as $x \to 2^+$ (*ie* from the right, $x = 2.1, 2.01, 2.001, \ldots$), the value of the function decreases to 3.
So we have $\lim_{x\to 2^-} f(x) = 3 = \lim_{x\to 2^+} f(x)$ and so $\lim_{x\to 2} f(x) = 3$.
We could have seen this numerically as well.

x	$f(x)$	x	$f(x)$
1.9	2.81	2.1	3.21
1.99	2.9801	2.01	3.0201
1.999	2.998001	2.001	3.002001

Can you see what these values are saying about $\lim_{x\to 2^-} f(x)$ *and* $\lim_{x\to 2^+} f(x)$ *and hence about* $\lim_{x\to 2} f(x)$ *?*

If we look at our example above, we see that $\lim_{x\to 2} f(x)$ is simply the value of $f(x)$ at $x = 2$. We can see that this must be the case because if we traced along the curve from either side of 2 towards 2, we would not experience any breaks in the graph and so we approach $f(2)$. This means that our function $f(x) = x^2 - 2x + 3$ is $\underline{\text{continuous at } x = 2}$.

We say that $\underline{f(x) \text{ is continuous at } x = a}$ if three conditions are met.
(i) $f(a)$ is defined
(ii) $\lim_{x\to a} f(x)$ exists
(iii) $\lim_{x\to a} f(x) = f(a)$

A function is continuous at $x = a$ if you can draw the graph near at $x = a$ without lifting your pencil. A function is called continuous if it is continuous for all x in its domain. If there is a place where there is a break in the graph, we have a discontinuity (and at least one of the three conditions above is violated).

Limits and Continuity

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and c is any constant, we have the following limit properties.
(i) $\lim_{x\to a} c = c$
(ii) $\lim_{x\to a} x = a$
(iii) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
(iv) $\lim_{x\to a}(cf(x)) = c\left(\lim_{x\to a} f(x)\right)$
(v) $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
(vi) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
(vii) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$, if n is rational
(viii) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, provided the root on the right hand side exists

These allow us to find many limits (which are really then being calculated by substitution).

Examples:
(i) $\lim_{x\to -3} 27 = 27$

(ii) $\lim\limits_{x\to -3} x = -3$

(iii) $\lim\limits_{x\to 2}(2x^3+3x-4) = 2\left(\lim\limits_{x\to 2} x\right)^3 + 3\left(\lim\limits_{x\to 2} x\right) - \lim\limits_{x\to 2} 4 = 2(2)^3+3(2)-4 = 18$

(iv) $\lim\limits_{x\to 1} \dfrac{5x}{x-1} = \dfrac{5\left(\lim_{x\to 1} x\right)}{\lim_{x\to 1}(x-1)} = \dfrac{5}{0}$ does not exist

(v) $\lim\limits_{x\to 4} \sqrt{\dfrac{x^2+1}{x+2}} = \sqrt{\dfrac{\left(\lim_{x\to 4} x\right)^2+1}{\left(\lim_{x\to 4} x\right)+2}} = \sqrt{\dfrac{4^2+1}{4+2}} = \sqrt{\dfrac{17}{6}}$

What we can recognize is that algebraic functions, rational functions and polynomials are all continuous everywhere on their domains and hence limits (within the domains) can be calculated by simply substituting.

Examples:

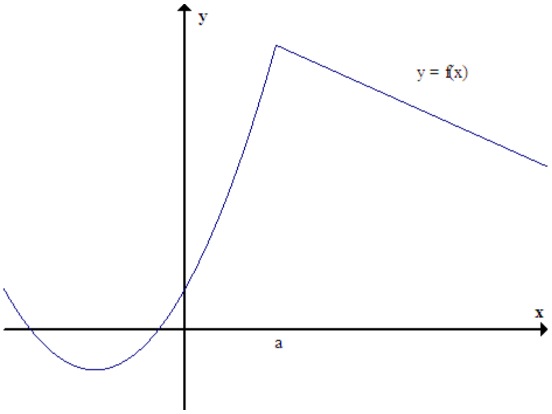
(i) $\lim\limits_{x\to -1}(2x^2+7x-2) = 2(-1)^2+7(-1)-2 = -7$

(ii) $\lim\limits_{x\to 3} \dfrac{x+7}{x-2} = \dfrac{3+7}{3-2} = \dfrac{10}{1} = 10$

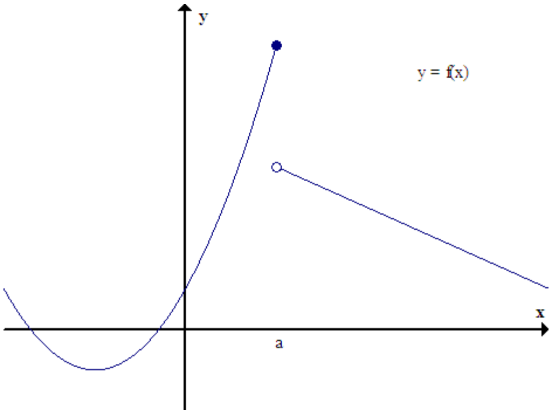
(iii) $\lim\limits_{x\to 2} \sqrt{x-7} = \sqrt{-5}$ does not exist (2 *is not in the domain*)

(iv) $\lim\limits_{x\to 2} \sqrt{2x+1} = \sqrt{2(2)+1} = \sqrt{5}$

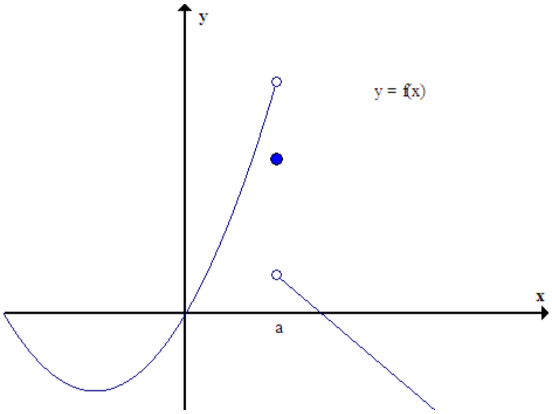
We said that $f(x)$ is continuous at $x=a$ if $f(a)$ is defined, $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} f(x) = f(a)$ and we said that this would correspond to being able to draw the graph of $f(x)$ without lifting our pencil or without there being a break in the graph at $x=a$. But what if there is a discontinuity at $x=a$? Consider the following graphs.



$f(a)$ defined, $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ so $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} f(x) = f(a)$ and thus $f(x)$ is continuous at $x = a$.

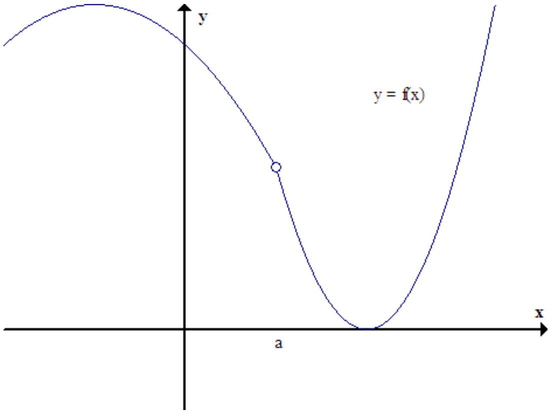


$f(a)$ is defined, $\lim_{x\to a^-} f(x)$ exists and $\lim_{x\to a^+} f(x)$ exists but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ so $\lim_{x\to a} f(x)$ does not exist and thus $f(x)$ is discontinuous at $x = a$. This is called a <u>jump discontinuity</u>.

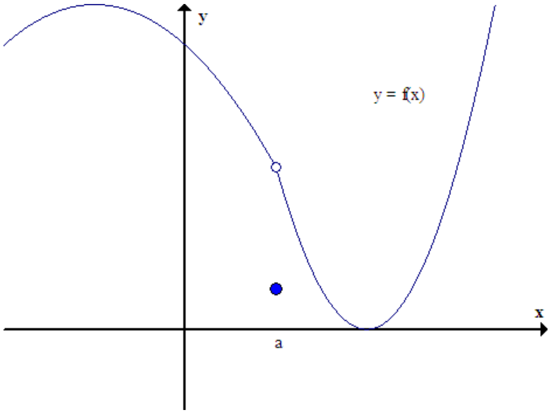


$f(a)$ is defined, $\lim_{x\to a^-} f(x)$ exists and $\lim_{x\to a^+} f(x)$ exists but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ so $\lim_{x\to a} f(x)$

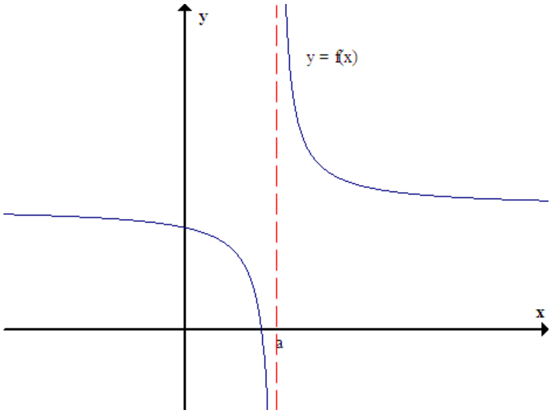
does not exist and thus $f(x)$ is discontinuous at $x = a$. This is called a jump discontinuity.



$f(a)$ is not defined, but $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ exists, but $f(x)$ is discontinuous at $x = a$. This is called a hole or removable discontinuity.



$f(a)$ is defined, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$, so $f(x)$ is discontinuous at $x = a$. This is called a hole or removable discontinuity.

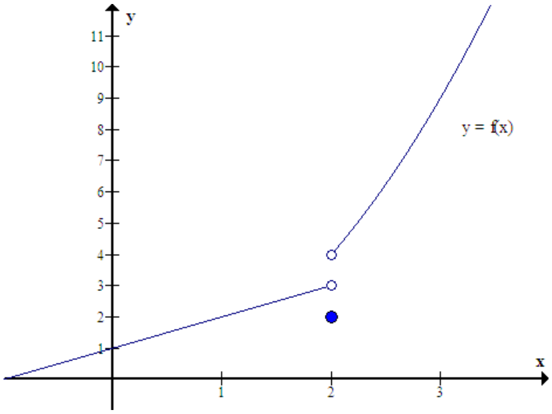


$f(a)$ is not defined, $\lim_{x \to a^-} f(x)$ does not exist, $\lim_{x \to a^+} f(x)$ does not exist and $\lim_{x \to a} f(x)$ does not exist. $f(x)$ is discontinuous at $x = a$. This is called a vertical asymptote.

Example:

Consider the function $f(x) = \begin{cases} 1 + x & x < 2 \\ 2 & x = 2 \\ x^2 & x > 2 \end{cases}$.

This is an example of a piecewise defined function.



$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1 + x) = 3$,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$,

so $\lim_{x \to 2} f(x)$ does not exist (so $f(x)$ cannot be continuous at $x = 2$),

while $f(2) = 2$,

but $f(x)$ has a jump discontinuity at $x = 2$.

Consider the following limits.

(i) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

(ii) $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$

(iii) $\lim\limits_{x\to 2} \dfrac{(x-1)^2-1}{x-2}$

Substituting into all of these will yield the <u>indeterminate form</u> $\frac{0}{0}$ (which is undefined). We can evaluate limits like these by doing certain manipulations.

(i) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

$= \lim\limits_{x\to 3} \dfrac{(x+3)(x-3)}{x-3}$ *(factor)*

$= \lim\limits_{x\to 3}(x+3)$ *(cancel common factor – allowed because $x \neq 3$)*

$= 6$

(ii) $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$

$= \lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1} \times \dfrac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$ *(rationalize the numerator)*

$= \lim\limits_{x\to 1} \dfrac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)}$

$= \lim\limits_{x\to 1} \dfrac{x-1}{(x-1)(\sqrt{x+3}+2)}$

$= \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x+3}+2}$ *(cancel common factor)*

$= 1/4$

(iii) $\lim\limits_{x\to 2} \dfrac{(x-1)^2-1}{x-2}$

$= \lim\limits_{x\to 2} \dfrac{x^2-2x+1-1}{x-2}$ *(expand)*

$= \lim\limits_{x\to 2} \dfrac{x^2-2x}{x-2}$

$= \lim\limits_{x\to 2} \dfrac{x(x-2)}{x-2}$ *(factor)*

$= \lim\limits_{x\to 2} x$ *(cancel common factor)*

$= 2$

(All of these discontinuities are removable.)

Example:

Is $f(x) = \dfrac{x^2 - 2x - 3}{x^2 + 5x + 4}$ continuous at $x = -1$? Does the limit exist at $x = -1$?

$f(-1) = \frac{0}{0}$, so $f(x)$ is not defined at $x = -1$ and so it cannot be continuous there.

$\lim\limits_{x \to -1} \dfrac{x^2 - 2x - 3}{x^2 + 5x + 4}$
$\lim\limits_{x \to -1} \dfrac{(x-3)(x+1)}{(x+1)(x+4)}$
$\lim\limits_{x \to -1} \dfrac{x-3}{x+4}$
$= -4/3$
so yes, $\lim\limits_{x \to -1} f(x)$ exists.
And this discontinuity is removable – so the graph of the function would have a hole at point $(-1, -4/3)$.

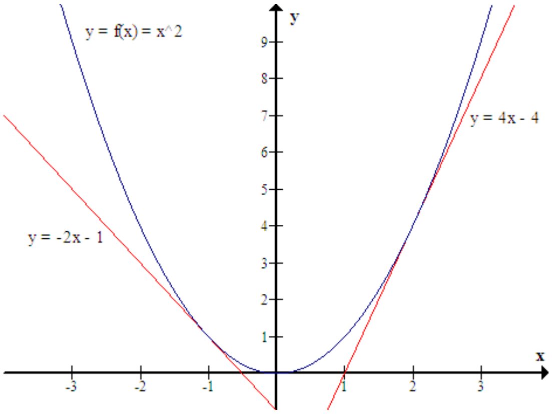
Introduction to Derviatives

If we go back to what we had previously, we had said that the instantaneous rate of change of a function $y = f(x)$ at point $P = (a, f(a))$, which is the slope m of the tangent line to the curve at that point, could be found by taking the slope of the secant line through P and $Q = (a + h, f(a + h))$, which would be the difference quotient $\dfrac{f(a+h) - f(a)}{h} = \dfrac{\Delta f}{\Delta x} = \dfrac{\Delta y}{\Delta x}$, and letting h shrink to 0. We can now understand that we mean take the limit as $h \to 0$. And so $m = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$. Note that this will always be an indeterminate form $\frac{0}{0}$ on substitution of $h = 0$, so we will have to do some manipulations to calculate these limits.

The instantaneous rate of change of $y = f(x)$ at $P = (a, f(a))$ is equal to the slope of the tangent line to the curve $y = f(x)$ at $x = a$ and is also called $\underline{\text{the derivative of } y = f(x) \text{ at } x = a}$ and is written $f'(a)$.
So $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ (which is called the $\underline{\text{first principles definition}}$).

Example:
Consider the function $y = f(x) = x^2$. Find the derivative and the equations of the tangent

lines at $x = -1$ and $x = 2$.



$$
\begin{aligned}
f'(-1) &= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}\\
&= \lim_{h\to 0}\frac{(-1+h)^2-(-1)^2}{h}\\
&= \lim_{h\to 0}\frac{(1-2h+h^2)-(1)}{h}\\
&= \lim_{h\to 0}\frac{-2h+h^2}{h}\\
&= \lim_{h\to 0} -2+h\\
&= -2
\end{aligned}
$$

so the tangent line passes through $(-1, 1)$ with slope $m = -2$ and so its equation is $y - y_0 = m(x - x_0)$ (*slope-point formula*) or $y - 1 = (-2)(x - (-1))$ or $y = -2x - 1$.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{(2+h)^2-(2)^2}{h}\\
&= \lim_{h\to 0}\frac{(4+4h+h^2)-(4)}{h}\\
&= \lim_{h\to 0}\frac{4h+h^2}{h}\\
&= \lim_{h\to 0} 4+h\\
&= 4
\end{aligned}
$$

so the tangent line is $y - 4 = (4)(x - 2)$ or $y = 4x - 4$.

If we look at the graph of $y = f(x) = x^2$, we can see that we would be able to draw a tangent line to the curve at any point but its slope would depend on the value of x. So the derivative of a function is itself a function of x and we can find it as such

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

So, for $f(x) = x^2$,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h}$

$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - (x^2)}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h}$

$= \lim_{h \to 0} 2x + h$

$= 2x$ *(can you see that this is what we have in the example above?)*

There is another notation for the derivative: if $y = f(x)$, $f'(x) = y'$ and $f'(x) = \frac{df}{dx} = \frac{dy}{dx}$ (from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$) and $f'(x) = \frac{d}{dx}(f(x))$ (operator notation). If we are evaluating the derivative at $x = a$, $f'(a) = y'(a) = \frac{dy}{dx}|_{x=a} = \frac{df}{dx}|_{x=a}$. $\frac{dy}{dx}$ is read as "dee y by dee x" and it represents the derivative of y with respect to (wrt) x.

Example:

If $f(x) = x^3$,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^3 - (x)^3}{h}$

$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - (x^3)}{h}$

$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$

$= \lim_{h \to 0} 3x^2 + 3xh + h^2$

$= 3x^2$.

Example:

If $f(x) = \frac{1}{x} = x^{-1}$,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x - (x+h)}{x(x+h)} \right]$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{x(x+h)} \right]$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$
$$= \frac{-1}{x^2}.$$

Do you notice something? We've just seen this:

$f(x)$	$f'(x)$
x^3	$3x^2$
x^2	$2x$
x^{-1}	$-x^{-2}$

Can you see a pattern?

What if $y = f(x) = mx + b$, a straight line? Then the line would be tangent to itself everywhere and hence the slope of the tangent would always be m. So we must have that $f'(x) = m$. Let's verify:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(m(x+h)+b) - (mx+b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h}$$
$$= \lim_{h\to 0} m$$
$$= m \quad \textit{(as expected)}.$$

Example:

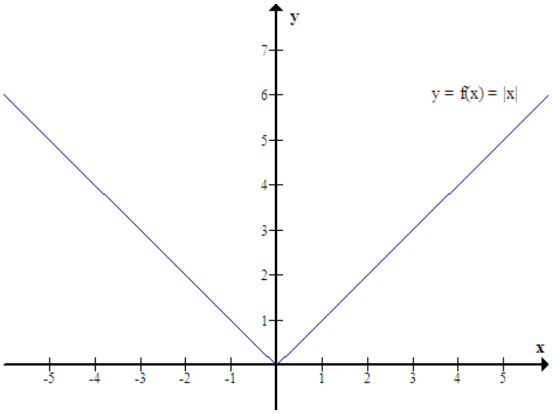
Let's go back to our moving object with position function $s(t) = 2t^2 + 3t + 2$ (where s is in m and t in s). The <u>velocity $v(t)$</u> of the object is the instantaneous rate of change of position with respect to time, or the derivative of $s(t)$ with respect to t.

So $v(t) = s'(t) = \frac{ds}{dt} = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h}$

$$= \lim_{h\to 0} \frac{(2(t+h)^2 + 3(t+h) + 2) - (2t^2 + 3t + 2)}{h}$$
$$= \lim_{h\to 0} \frac{(2(t^2 + 2th + h^2) + 3t + 3h + 2) - (2t^2 + 3t + 2)}{h}$$
$$= \lim_{h\to 0} \frac{4th + 2h^2 + 3h}{h}$$
$$= \lim_{h\to 0} 4t + 2h + 3$$
$$= 4t + 3$$

so, in particular, the velocity at time $t = 2$ s is $v(2) = s'(2) = 4(2) + 3 = 11$ m/s.

Since the derivative (or instantaneous rate of change or slope of the tangent line) is defined by a limit, we should be careful to realize that it does not have to exist for all x. We say that $f(x)$ is <u>differentiable at $x = a$</u> if $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists. How could $f'(a)$ fail to exist? If a is not in the domain of f, $f(a)$ is undefined and $f'(a)$ cannot exist. If f is discontinuous at $x = a$, $f'(a)$ will also fail to exist (*we can't draw a tangent to the curve at* *that point*). But, it is also possible for $f(x)$ to be continuous at $x = a$ and for $f'(a)$ to fail to exist. Consider $y = f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$.



Notice that $f(x)$ is continuous at $x = 0$ as $f(0) = 0$ and $\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x)$. But, if we try to find $f'(0)$, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$, we'll have to consider the two one-sided limits separately as the definition of the function is different for positive and negative values.
$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1.$
And $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$
Thus $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$ and so $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist. And hence, $y = f(x) = |x|$ is not differentiable at 0. The graph of $y = f(x) = |x|$ is said to have a corner at $x = 0$.

$f'(a)$ will also fail to exist if there is a cusp or vertical tangent at $x = a$.