University of Ottawa

Department of Mathematics and Statistics

MAT 1302C : Mathematical Methods II

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Second Midterm Exam – Version A

November 22, 2011

Surname ____________________ First Name ____________________

Student # ____________________ DGD (Tuesday/Thursday) ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	5	4	4	3	4	24
Grade							

1. (a) **(3 points)** Use cofactor expansion across a row to compute the determinant of the following matrix. (You can choose any row that you want.)

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & 3 & 5 & 1 \\ 3 & -1 & 1 & 2 \end{bmatrix}$$

Solution: It would be most convenient to expand across the second row:

$$\begin{vmatrix} 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & 3 & 5 & 1 \\ 3 & -1 & 1 & 2 \end{vmatrix} = -2 \begin{vmatrix} 0 & -2 & 1 \\ 3 & 5 & 1 \\ -1 & 1 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 & -2 \\ 1 & 3 & 5 \\ 3 & -1 & 1 \end{vmatrix}$$

expansion across the first row: $\begin{vmatrix} 0 & -2 & 1 \\ 3 & 5 & 1 \\ -1 & 1 & 2 \end{vmatrix} = -(-2) \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 5 \\ -1 & 1 \end{vmatrix} = 2(7) + 8 = 22$

expansion along the first row: $\begin{vmatrix} 1 & 0 & -2 \\ 1 & 3 & 5 \\ 3 & -1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 3 & 5 \\ -1 & 1 \end{vmatrix} + (-2) \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = 8 + (-2)(-10) = 28$

Therefore $\det A = -44 - 28 = -72$.

(b) **(1 point)** Is A invertible? Justify your answer.

Solution: Yes, because $\det A \neq 0$.

2. (a) **(3 points)** Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 4 & -\frac{1}{2} & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{4}{5} \\ 0 & 0 & -1 \end{bmatrix}$. Suppose that a 3×3 matrix C satisfies

$$B^{-1}A(C^T)^{-1}B^2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute $\det C$. Justify your answer.

Solution: Since A and B are triangular, we have $\det A = -3$ and $\det B = \frac{5}{2}$. We now have

$$-4 = \det(B^{-1}A(C^T)^{-1}B^2) = (\det B)^{-1}\det(A)\det(C)^{-1}\det(B)^2 = (\frac{2}{5})(-3)(\det C)^{-1}(\frac{5}{2})^2$$

from which it follows that $\det C = \dfrac{15}{8}$.

(b) **(2 points)** Suppose a, b, c are three numbers such that $\begin{vmatrix} a & 1 & 3 & 1 \\ b & 0 & -2 & \frac{1}{3} \\ -2 & 3 & 5 & 1 \\ c & \frac{1}{2} & -5 & 2 \end{vmatrix} = 5$. What is the value of $\begin{vmatrix} a+2 & 1 & 3 & 3 \\ b & 0 & -2 & 1 \\ 4 & 3 & 5 & 3 \\ c+1 & \frac{1}{2} & -5 & 6 \end{vmatrix}$? Justify your answer.

Solution: The second matrix can be obtained from the first one by the following column operations:

$$\begin{vmatrix} a & 1 & 3 & 1 \\ b & 0 & -2 & \frac{1}{3} \\ -2 & 3 & 5 & 1 \\ c & \frac{1}{2} & -5 & 2 \end{vmatrix} \xrightarrow[C_4 \to 3C_4]{C_1 \to C_1 + 2C_2} \begin{vmatrix} a+2 & 1 & 3 & 3 \\ b & 0 & -2 & 1 \\ 4 & 3 & 5 & 3 \\ c+1 & \frac{1}{2} & -5 & 6 \end{vmatrix}$$

The replacement doesn't change the determinant whereas scaling the last column by $c = 3$ multiplies the determinant by c. Therefore $\begin{vmatrix} a+2 & 1 & 3 & 3 \\ b & 0 & -2 & 1 \\ 4 & 3 & 5 & 3 \\ c+1 & \frac{1}{2} & -5 & 6 \end{vmatrix} = 15$.

3. Consider the matrix

$$A = \begin{bmatrix} 3 & -6 & 0 & 4 & 0 \\ 1 & -2 & 1 & 5 & -1 \\ 1 & -2 & 0 & 2 & 0 \end{bmatrix}.$$

(a) **(3 points)** Find a basis for Nul A.

Solution: To solve the homogeneous system $A\mathbf{x} = \mathbf{0}$ we row reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccccc|c} 3 & -6 & 0 & 4 & 0 & 0 \\ 1 & -2 & 1 & 5 & -1 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & \frac{4}{3} & 0 & 0 \\ 1 & -2 & 1 & 5 & -1 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{11}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{3}{2}R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{11}{3} & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - \frac{11}{3}R_3]{R_1 \to R_1 - \frac{4}{3}R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \text{R.E.F.} \Rightarrow \begin{cases} x_1 = 2x_2 \\ x_2 : \text{free} \\ x_3 = x_5 \\ x_4 = 0 \\ x_5 : \text{free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{basis for Nul } A : \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) **(1 point)** Determine the rank of A. Justify your answer.

Solution: By the Rank Theorem, rank $A = 5 - 2 = 3$.

4. **(4 points)** Determine if the following matrix is invertible. If it is invertible, then find its inverse.

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 3R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 2 & 6 \\ 0 & 1 & 0 & -4 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

It follows that the above matrix is invertible and its inverse is $\begin{bmatrix} -7 & 2 & 6 \\ -4 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

5. Determine if the following subsets of $\mathbb{R}^3$ are subspaces or not. Justify your answers.

(a) **(1 point)** $H = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \;\middle|\; x, y \text{ are in } \mathbb{R} \right\}$.

Solution: This is not a subspace because it does not contain the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) **(1 point)** $H = \left\{ \begin{bmatrix} x+2y \\ -3y \\ x-2y \end{bmatrix} \;\middle|\; x, y \text{ are in } \mathbb{R} \right\}$.

Solution: This is a subspace because $\begin{bmatrix} x+2y \\ -3y \\ x-2y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ and since x, y are arbitrary, we have $H = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \right\}$ which implies that H is a subspace.

(c) **(1 point)** $H = \left\{ \begin{bmatrix} x \\ y \\ z^2 \end{bmatrix} \;\middle|\; x, y, z \text{ are in } \mathbb{R} \right\}$. **Solution:** This is not a subspace because for example $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is in H but $(-1)\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ is not in H, as $z^2 \geq 0$ for every real number z.

6. An economy has two sectors: Manufacturing and Service. In order to produce one unit of output, Manufacturing requires .3 units from its own sector and .6 units from Service. On the other hand, Service requires .4 units from its own sector and .2 units from Manufacturing to produce one unit of output.

(a) **(1 point)** Write down the consumption matrix C for this economy.

Solution:
If Manufacturing is the first sector and Service is the second sector, then

$$C = \begin{bmatrix} .3 & .2 \\ .6 & .4 \end{bmatrix}$$

(b) **(1 point)** Determine the intermediate demands if Service decides to produce 15 units and Manufacturing decides to produce 20 units.

Solution:
The intermediate demand will be

$$15 \begin{bmatrix} .2 \\ .4 \end{bmatrix} + 20 \begin{bmatrix} .3 \\ .6 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \end{bmatrix}$$

The intermediate demand is 9 units from Manufacturing and 18 units from Service.

(c) **(2 points)** Determine the production levels required to meet a final demand of 15 units from Manufacturing and 9 units from Service.

Solution: We should solve the Leontief equation $(I - C)\mathbf{x} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$. We have

$$I - C = \begin{bmatrix} .7 & -.2 \\ -.6 & .6 \end{bmatrix} \quad \Longrightarrow \quad (\mathrm{I} - C)^{-1} = \frac{1}{.3} \begin{bmatrix} .6 & .2 \\ .6 & .7 \end{bmatrix} = \begin{bmatrix} 2 & \frac{2}{3} \\ 2 & \frac{7}{3} \end{bmatrix}$$

$$\Longrightarrow \mathbf{x} = (\mathrm{I} - C)^{-1} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 & \frac{2}{3} \\ 2 & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 36 \\ 51 \end{bmatrix}$$