MATH 223 - EXAM #1
OCTOBER 9, 2013

Name: Solutions.
Student ID:

Exam rules:

- No calculators, open books or notes are allowed.
- You do not need to prove results that we proved in class or that appeared in the homework.
- There are 4 problems in this exam. Each problem is worth 5 marks.
- All vector spaces are over real numbers. The notation is the usual one:
  - $\mathbb{R}^n$ – the real n-space.
  - $M_{m\times n}$ – the space of $m \times n$ matrices.
  - $Sym_n, Skew_n$ – the symmetric and skew-symmetric matrices.
  - P_n – the space of polynomials of degree at most n

PROBLEM 1. Let $T : V \to W$ be a linear transformation and let $Z \subset V$ be a subspace. Show that the set

$$U = \{\text{All vectors } \vec{w} \text{ in } W, \text{ such that } \vec{w} = T(\vec{z}) \text{ for some } \vec{z} \text{ in } Z\}$$

is a subspace of W. In your solution, indicate clearly where you use the facts that T is linear and Z is a subspace.

Check the 3 properties of a subspace

1) $\vec{0}$ is in U because $\vec{0} = T(\vec{0})$ and $\vec{z} = \vec{0}$ lies in Z.

2) Assume $\vec{u}_1$ and $\vec{u}_2$ are in U, $\vec{u}_1 = T(\vec{z}_1)$, $\vec{u}_2 = T(\vec{z}_2)$.
Then $\vec{u}_1 + \vec{u}_2$ also lies in U because
$\vec{u}_1 + \vec{u}_2 = T(\vec{z}_1 + \vec{z}_2)$, $\vec{z}_1 + \vec{z}_2$ is in Z.

3) Assume $\vec{u} = T(\vec{z})$ lies in U. Then
$c \cdot \vec{u} = T(c \cdot \vec{z})$
also lies in U.

PROBLEM 2. Let $V \subset \mathbb{R}^4$ be the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ satisfying the following three equations:

$$x_1 - x_4 = 0$$
$$x_1 - x_2 + x_3 - x_4 = 0$$
$$x_2 - x_3 = 0$$

Find a basis for V. (You may assume without proof that V is a subspace.)

Equation 2 follows from equations 1 and 3.

A general vector in V has the form

$$\begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ form a basis for V.

PROBLEM 3. Consider the linear transformation

$$T : Sym_3 \to M_{2\times 3}$$

that takes a 3×3 symmetric matrix and removes its last row. Find the rank and nullity of T. (You may assume that T is linear.)

T maps

$$T : \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \longmapsto \begin{bmatrix} a & b & c \\ b & d & e \end{bmatrix}$$

The nullspace consists of all matrices where

$$a = b = c = d = e = 0$$

These are:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f \end{bmatrix}.$$

This space has dimension 1. $= \dim N(T)$

From dimension theorem

$$\dim N(T) + \dim R(T) = \dim Sym_3$$

$$\dim N(T) = 1, \quad \dim Sym_3 = 6$$

From this, $\dim R(T) = 5$.

PROBLEM 4. Let $V \subset P_{42}$ be the subspace of all polynomials $p(x)$ such that $p(1) = 0$. Does the set

$$S = \{1 - x, x - x^2, x^2 - x^3, \ldots, x^{41} - x^{42}\}$$

span V? If it does span, prove it. If not, explain why. (You may assume that V is a subspace)

The space V has

$$\dim V = \dim P_{42} - 1 = 42.$$

To prove this, consider the linear transformation

$$T : P_{42} \longrightarrow \mathbb{R}$$
$$p(x) \longmapsto p(1).$$

Then $V = N(T)$. Since T is onto, the dimension theorem says

$$\underbrace{\dim N(T)}_{\dim V} + \underbrace{\dim R(T)}_{1} = \dim P_{42}$$

The set S has 42 elements, hence S spans V if and only if S is linearly independent. S is linearly independent because all polynomials have different degrees. So, S spans V.