

Applied Ordinary Differential Equations
ENGR 213 - Section F
 Prof. Alina Stancu

Exam I (A)

- (1) (10 points) Solve the initial value problem

$$x \frac{dy}{dx} + (x - 4)y = x^5, \quad y(1) = 1.$$

Solution: An integrating factor is

$$\mu(x) = e^{\int (1 - \frac{4}{x}) dx} = e^{x - 4 \ln|x|} = e^x \cdot e^{\ln x^{-4}} = e^x \cdot x^{-4} = \frac{e^x}{x^4}.$$

Multiplying both sides of the ODE with this integrating factor leads to

$$\left(\frac{e^x}{x^4} y \right)' = e^x.$$

Integrating both sides with respect to x , we obtain

$$\frac{e^x}{x^4} y = e^x + C \Rightarrow y = x^4 + C \frac{x^4}{e^x}, \quad C = \text{constant}.$$

We use $y(1) = 1$ to find the constant C as $y(1) = 1 + C/e = 1 \Rightarrow C = 0$, thus the solution to the IVP is $y(x) = x^4$, $x > 0$.

- (2) (10 points) Solve the given homogeneous equation by the appropriate substitution

$$\frac{dy}{dx} = \frac{3xy}{3x^2 + y^2}.$$

You may leave the solution in implicit form.

Solution: We use the standard substitution for homogeneous ODEs of first order, namely $u = y/x$ or $y = ux$ and, thus, $\frac{dy}{dx} = x \frac{du}{dx} + u$.

Therefore the ODE becomes

$$x \frac{du}{dx} + u = \frac{3u}{3 + u^2} \Rightarrow x \frac{du}{dx} = -\frac{u^3}{3 + u^2}.$$

The latter is a separable ODE $-\frac{3+u^2}{u^3} du = \frac{1}{x} dx$, or $-\left(\frac{3}{u^3} + \frac{u^2}{u^3}\right) du = \frac{1}{x} dx$. Integrating, we obtain $\frac{3}{2}u^{-2} - \ln|u| = \ln|x| + C$, where C is an arbitrary constant.

Hence we obtain the solution $y(x)$, in implicit form, of the original ODE

$$\frac{3x^2}{2y^2} - \ln|y/x| = \ln|x| + C, \quad C = \text{constant}.$$

- (3) (10 points) Solve the exact ordinary differential equation

$$(3 - 2 \sin(2x + 3y)) dx + (2 - 3 \sin(2x + 3y)) dy = 0,$$

leaving the solution in implicit form.

Solution: We seek a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = 3 - 2 \sin(2x + 3y), \quad \frac{\partial \phi}{\partial y} = 2 - 3 \sin(2x + 3y).$$

Integrating the first equation with respect to x , we obtain $\phi(x, y) = 3x + \cos(2x + 3y) + c(y)$. Consequently,

$$\frac{\partial \phi}{\partial y} = -3 \sin(2x + 3y) + c'(y) \Rightarrow c'(y) = 2 \quad c(y) = 2y + C, \quad C = \text{constant}.$$

It follows that the solution to the exact ODE (given in implicit form) is

$$3x + \cos(2x + 3y) + 2y = \tilde{C}, \quad \tilde{C} = \text{constant}.$$

- (4) (10 points) A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 100°F?

Solution: Newton's model of cooling tells us that the ODE describing the temperature of the cake $T(t)$ is

$$\frac{dT}{dt} = -K(T - 70), \quad T(0) = 210, \quad T(30) = 140, \quad K = \text{a constant to be determined}.$$

We solve first for the general solution of the ODE (which we treat as separable):

$$\frac{1}{T - 70} dT = -K dt \Rightarrow \ln|T - 70| = -Kt + C, \quad C = \text{constant}.$$

Using $T(0) = 210$ and $T(30) = 140$, we obtain $C = \ln 140$ and $\ln 70 = -30K + \ln 140 \Rightarrow 30K = \ln 2 \Rightarrow K = (\ln 2)/30$.

Thus

$$T = 70 + e^{-Kt+C} = 70 + e^{-t(\ln 2)/30 + \ln 140} = 70 + 140 \left(\frac{1}{2}\right)^{t/30}$$

and

$$100 = 70 + 140 \left(\frac{1}{2}\right)^{t/30} \Rightarrow t = 30 \ln(3/14) / \ln(1/2) = 30 \ln(14/3) / \ln 2 \approx 66.67 \text{ min}.$$