MAT 1320 Section Fall 2004

Professor:

Midterm Test 1: September 29, 2004

LAST Name:____

First Name:____

Student Number:____________________________

Instructions:

•This midterm exam has 8 problems and a total of 5 pages, including this cover page. You have 80 minutes.

•Only non-graphing calculators (TI30xx) are permitted. Notes and books are not permitted.

•Each problem is worth 2 marks.

•Problems 1 - 4 are multiple choice. Circle the letter corresponding to the correct answer; no part marks.

•Problems 5 - 8 require a detailed and clearly presented solution. Please read each question carefully and be sure to answer all that is asked.

•Work all problems in the space provided. Use the backs of the pages for rough work if necessary. If you continue your answer on the back of a page, indicate this clearly. Do not use any other paper. Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Work on multiple choice problems will be examined only in case of suspected fraud.

Problem 1. (2 points) Which of the following represents the Quotient Rule ?

A: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)}{g'(x)}$

B: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) + f(x)g'(x)}{(g(x))^2}$

C: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)}$

D: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

E: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x) - g'(x)}{(g(x))^2}$

F: none of these

Solution: D

Problem 2. (2 points) A table of values for f, f', g and g' is given.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	10	-1	5	3
3	1	2	2	7
5	-3	4	-2	4

If $h(x) = f(g(x))$, what is $h'(3)$?

A: 7 **B**: -7 **C**: 14 **D**: 10 **E**: -14 **F**: -10

Solution: Since $\frac{d}{dx}h(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$, we have $h'(3) = f'(g(3))g'(3) = f'(2)g'(3) = (-1)(7) = -7$, using the table of values. So the correct answer is **B**.

Problem 3. (2 points) Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality $\cos x \geq 1/\sqrt{2}$.

A: $[0, \pi/4] \cup [7\pi/4, 2\pi]$

B: no solution

C: $[\pi/4, 7\pi/4]$

D: $[0, \pi/2] \cup [3\pi/2, 2\pi]$

E: $[0, \pi/4]$

F: $[\pi/4, \pi/2] \cup [3\pi/2, 7\pi/4]$

Solution: Using the graph of $y = \cos(x)$, or drawing a circle and thinking of cosine as the horizontal coordinate of a point on the circle, we see that this equality will hold when x is near 0 or when x is near 2π, where $\cos(x) = 1$. More precisely, we note that $\cos(\pi/4) = \cos(7\pi/4) = 1\sqrt{2}$, and so the correct region is given by **A**.

Problem 4. (2 points) Which of the following is a correct formula for the inverse of the function $y = 2e^{x+5}$?

A: $y = \frac{1}{2}e^{-(x+5)}$ **B**: $y = -2e^{x+5}$ **C**: $y = \frac{1}{2}e^{x-5}$

D: $y = \ln(\frac{1}{2}x + 5)$ **E**: $y = \ln(\frac{1}{2}x) - 5$ **F**: $y = \ln(2x - 5)$

Solution: To find the inverse function, we solve for y in terms of x, and then swap the names of the variables. We have $y/2 = e^{x+5}$; apply ln to both sides and simplify to get $\ln(y/2) = x + 5$, so $x = \ln(y/2) - 5$. Swapping the names of the variables yields answer **E**.

Problem 5. (2 points) Find the equation of the tangent line to the curve $y = x\sin(2x)$ at the point $(2, -1.5136)$. Give the numbers in the answer to 4 decimal places and put the final answer in the form $y = mx + b$.

Solution: To find the equation of the tangent line, all we need its slope m; we are already given a point on the curve.

The slope of the tangent line at $x = 2$ is equal to the value of the derivative at $x = 2$.
We compute, using the product rule, and then the chain rule, that

$$y' = \sin(2x) + x\frac{d}{dx}(\sin(2x)) = \sin(2x) + x(\cos(2x)\cdot 2) = \sin(2x) + 2x\cos(2x).$$

Now plug in $x = 2$ (radians!), and we get $y' = m = -3.3714$.

The equation of a line is $y = mx + b$; plug in $(x, y) = (2, -1.5136)$ and $m = -3.3714$ to get $b = y - mx = -1.5136 - (-6.7428) = 5.2292$. Final answer:

$$y = -3.3714x + 5.2292$$

Marking scheme: Possible scores: $\{0, 1, 2\}$
1 point for correct evaluation of the derivative
1 point for equation based on their derivative
Example: student gets $y' = 2\cos(2x)$ (incorrect) BUT subsequently correctly calculates $m = 2\cos(4) = -1.3073$ and final answer $y = -1.3073x + 1.1010$. Mark = 1.

Problem 6. (2 points) If $f(x) = \cot(x)$, find the second derivative $f''(x)$. Write clearly and put a box around your final answer.

Solution: First form: We recall that if $f(x) = \cot(x)$, then $f'(x) = -\csc^2(x)$. Thus by the chain rule (or use the product rule),

$$f''(x) = -2\csc(x)(-\csc(x)\cot(x)) = 2\csc^2(x)\cot(x)$$

Second form: $f(x) = \cot(x) = \cos(x)/\sin(x)$, so by the quotient rule, $f'(x) = (-\sin^2(x) - \cos^2(x))/\sin^2(x) = -1/\sin^2(x)$. Again by the quotient rule,

$$f''(x) = -\frac{-(2\sin(x)\cos(x))}{\sin^4(x)} = \frac{2\cos(x)}{\sin^3(x)}$$

Marking scheme: Possible scores: $\{0, 0.5, 1, 1.5, 2\}$
0.5 points for correct $f'(x)$
1 point for correct use of chain rule, product rule or quotient rule to calculate $f''(x)$
0.5 points for correct algebra, correct recall of derivatives
Exception: student forgets one or both -1 signs in derivatives: deduct only 0.5
Examples:
student has $f'(x) = \csc(x)$ (wrong) but $f''(x) = -\csc(x)\cot(x)$: Mark 1.5
student has $f'(x) = \csc^2(x)$ (missing -1) and $f''(x) = 2\csc^2(x)\cot(x)$: Mark 1.5
student has $f'(x) = \csc^2(x)$ (missing -1) and $f''(x) = -2\csc^2(x)\cot(x)$: Mark 1.5
student has $f'(x) = -\csc^2(x)$ but $f''(x) = \cot^4(x)$ (many errors): Mark 0.5
student uses second form above, but leaves common factor of $\sin(x)$ in $f''(x)$: Mark 1.5

Problem 7. (2 points) Find the derivative of $f(x) = e^{x^2 \sin(x)}$.

Solution: Differentiating this function requires both the chain rule and the product rule:

$$\begin{aligned} f'(x) &= e^{x^2 \sin(x)} \frac{d}{dx}(x^2 \sin(x)) && \text{chain rule} \\ &= e^{x^2 \sin(x)}(2x \sin(x) + x^2 \cos(x)) && \text{product rule} \end{aligned}$$

Marking scheme: Possible scores: $\{0, 1, 2\}$
1 point for having applied first the chain rule (possibly with incorrect derivative of $x^2 \sin(x)$)
1 point for correct final answer
Examples:
student gives $e^{2x \sin(x) + x^2 \cos(x)}$: Mark 0 (chain rule was not applied)
student gives $e^{x^2 \sin(x)}(2x \cos(x))$ or $e^{x^2 \sin(x)}(2x \sin(x) - x^2 \cos(x))$: Mark 1

Problem 8. (2 points) Define the function $\arcsin(x)$, remembering to state its domain and range. Sketch the graph of $y = \arcsin(x)$, and label endpoints with their coordinates.

Solution: We define $y = \arcsin(x)$ if $\sin(y) = x$ and $-\pi/2 \leq y \leq \pi/2$. Thus the range of the function is $[-\pi/2, \pi/2]$ and the domain is $[-1, 1]$.
The graph is in your notes.

Marking scheme: Possible scores: $\{0, 1, 1.5, 2\}$
1 point for $\sin(y) = x$ with correct domain and range
1 point for graph with correctly labeled axes and passing through the points $(0, 0)$, $(-1, -\pi/2)$, and $(1, \pi/2)$
Exception: Some students will have the domain and range correct in the definition, but not on the graph, or vice versa; in this case, if everything else is fine, then the mark is 1.5.
Example: Definition given is $y = \csc(x)$ or $y = 1/\sin(x)$: Mark = 0, no matter what the graph looks like.
Definition given is $y = \sin(x)$: Mark = 0 or 1, depending on graph.