[**1**] 1. For what value of α is the set of vectors $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ \alpha \end{pmatrix} \right\}$ linearly dependent?

Choices:

A. -1

B. 2

C. 0

D. 1

E. -1/2

F. -2

Solution: Applying elementary row operations

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 0 & \alpha \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & \alpha - 1 \end{pmatrix}$$

where the second matrix is in REF, so we can conclude that for $\alpha = 1$ the matrix will have rank 2. Thus, the correct answer is **D**.

[**1**] 2. If A is a 7×12 matrix, what is the smallest possible dimension of the kernel of A?

Choices:

A. 0

B. 3

C. 5

D. 7

E. 11

F. 12

Solution: In this case, $\ker A = \text{Null}(A)$. Now, we know

$$\begin{aligned} n &= \dim \text{Null}(A) + \text{rank}(A) \\ 12 &= \text{Null}(A) = \text{rank}(A) \end{aligned}$$

so

$$\text{Null}(A) = 12 - \text{rank}(A).$$

We also have that $\text{rank}(A)$ is less or equal to the number of columns, which is 7. Therefore,

$$\text{Null}(A) = 12 - \text{rank}(A) \geq 12 - 7 = 5.$$

Thus, the correct answer is **C**.

[1] 3. If $B = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$, then the second row of B^{-1} is:

Choices:

A. $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$

B. $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$

C. $\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$

D. $\begin{pmatrix} 2 & 0 & -1 \end{pmatrix}$

E. $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$

F. None of the above.

Solution: A direct computation shows

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right).$$

Thus, the correct answer is **B**.

SHOW ALL YOUR WORK AND JUSTIFY YOUR ANSWERS!

[6] 4. Let $W = \text{Span}\left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\1\\4\\2\end{pmatrix}, \begin{pmatrix}0\\1\\3\\1\end{pmatrix}\right\}$, and define a matrix A by

$$A = \begin{pmatrix}1 & -1 & 1 & 0\\0 & 1 & 1 & 1\\1 & 2 & 4 & 3\\1 & 0 & 2 & 1\end{pmatrix}.$$

a) Find a basis for W which is a **<u>subset</u>** of the given spanning set.

b) Extend your basis in part (a) to a basis of $\mathbb{R}^4$.

c) Find a basis for Null(A) and hence find its dimension.

d) Extend your basis of Null(A) to a basis of $\mathbb{R}^4$.

Solution: We have that

$$\begin{pmatrix}1 & -1 & 1 & 0\\0 & 1 & 1 & 1\\1 & 2 & 4 & 3\\1 & 0 & 2 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 2 & 1\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}.$$

a) Form the computation above we have that the first columns of the matrix on the right are the only with a leading one in it, so the first two columns from the matrix on the left are a basis of W. Thus, an answer is

$$\left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\0\end{pmatrix}\right\}.$$

b) We could have answer this question using the standard method saw in class, or using results above the row space and null space of a matrix. But instead, using that the second component of the first vector in the basis and the last component of the second vector in the basis are both zero, the corresponding standard unit vectors will not be linear combination of them:

$$\left(\begin{array}{cc|cc}1 & -1 & 0 & 0\\0 & 1 & 1 & 0\\1 & 2 & 0 & 0\\1 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{array}\right).$$

Thus,

$$\left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

is a basis of $\mathbb{R}^4$.

c) From the computation done at the beginning we have that the solution to the homogeneous system $A\vec{x} = \vec{0}$ is

$$x_1 = -2x_3 - x_4, \qquad x_2 = -x_3 - x_4.$$

Set $x_3 = r$, and $x_4 = s$. Thus, the solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2r - s \\ -r - s \\ r \\ s \end{pmatrix} = r \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, a basis of Null(A) is

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

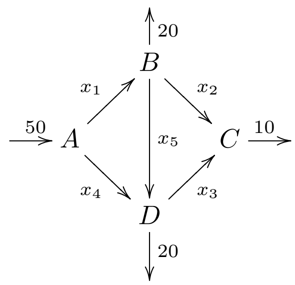
d) Since Row(A) is an orthogonal complement of Null(A), we have that

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^4$.

SHOW ALL YOUR WORK AND JUSTIFY YOUR ANSWERS!

[6] 5. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter A or leave B,C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \dots, 5$. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 1 & 1 & 30 \\ 0 & 0 & 1 & -1 & -1 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Give the general solution. (Ignore the constraints from (a) at this point.)

c) If $\overline{BD}$ is closed find the maximum and minimum flows along $\overline{BC}$

Solution:

a)

	IN	OUT
A	50	$x_1 + x_4$
B	x_1	$x_2 + x_5 + 20$
C	$x_2 + x_3$	10
D	$x_4 + x_5$	$x_3 + 20$

Constraints: All the x_i's are integers and $x_i \geq 0$.

b) From the matrix we have that x_4 and x_5 are the independent variables. Set $x_4 = r$, and $x_5 = s$. Then, the solution to the system is

$$x_1 = 50, \quad x_2 = 30 - r - s, \quad x_3 = -20 + r + s, \quad x_4 = r, \quad x_5 = s.$$

Its parametric equation is then

$$\begin{pmatrix} 50 \\ 30 \\ -20 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

c) If $\overline{BD} = x_5$ is closed, means that $x_5 = s = 0$. So, when $s = 0$ we have that $\overline{BC} = x_2 = 30 - r$, and

$$50 \geq r, \qquad 30 \geq r, \qquad r \geq 20.$$

In other words, we must have $30 \geq r \geq 20$. Thus, if $r = 30$ then

$$x_2 = 0,$$

and if $r = 20$ then

$$x_2 = 10.$$

Thus, the maximum flow at x_2 is 10, and the minimum flow at x_2 is zero.

SHOW ALL YOUR WORK AND JUSTIFY YOUR ANSWERS!

[6] 6. Suppose A is an $n \times n$ matrix and that there is a vector $b \in \mathbb{R}^n$, for which $Ax = b$ is inconsistent.

In the box after the statement state whether each of the following is (always) true, or (possibly) false.

- If you say the statement may be false, you must give an explicit example (with numbers!).
- If you say the statment is true, you must give a clear explanation.

a) The system $Ax = 0$ has a unique solution. | *FALSE*

Solution: Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b) The matrix A is not invertible. | *TRUE*

Solution: Recall that $Ax = b$ consistent if and only if A is invertible.
Thus, $Ax = b$ inconsistent if and only if A is not invertible.

c) The rows of A are linearly independent. | *FALSE*

Solution: Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

SHOW ALL YOUR WORK AND JUSTIFY YOUR ANSWERS!

7. Bonus]

a) Let A be an invertible matrix. Show that A^{-1} is unique. **Solution:** Assume that B_1 and B_2 are two inverses of A. Then

$$B_1 = IB_1 = (B_2A)B_1 = B_2(Ab_1) = B_2I = B_2.$$

b) Let $V \subset \mathbb{R}^n$ be a subspace. Recall that $V^\perp$ is the subspace of $\mathbb{R}^n$ consisting of all the vector orthogonal to every vector in V.

Let A be an $m \times n$ matrix. Show that $(\mathrm{Col}(A))^\perp = \mathrm{Null}\left(A^T\right)$.

Solution: We know that $\mathrm{Null}(A) = (\mathrm{Row}(A))^\perp$.

We also know that $\mathrm{Col}(A) = \mathrm{Row}(A^T)$.

Thus

$$(\mathrm{Col}(A))^\perp = \left(\mathrm{Row}(A^T)\right)^\perp = \mathrm{Null}(A^T).$$