Sample Midterm test - Math 205

1. Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using 50 approximating rectangles, right endpoints and $\sum -$ notation (do not evaluate). Is this approximation an underestimate or an overestimate?

2. Determine the function $f(x)$ if:

$$f(x) = \frac{d}{dx} \int_{x^2}^{0} \sqrt{1+t^3}\,dt.$$

3. Evaluate:

   (a) $\displaystyle\int \frac{x\cos\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}dx$ by using proper substitution;

   (b) $\displaystyle\int_0^2 \sqrt{x^3+1}\,x^5 dx$ by using substitution $u = x^3 + 1$

4. Use the method by parts to evaluate: $\displaystyle\int_1^e (\ln x)^2\,dx$.

5. Show that $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$.

6. Use the formula $I_n = \displaystyle\int \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$ to calculate

   (a) $\displaystyle\int \cos^4 x dx$ and

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{10} x dx$.

7. Use partial fractions to evaluate the following integrals:

   (a) $\displaystyle\int \frac{3x dx}{(x+3)(x-2)}$

   (b) $\displaystyle\int_1^{\sqrt{3}} \frac{x^2}{x^2+1}dx$

8. Evaluate $\displaystyle\int \frac{2x - x^3}{\sqrt{4-x^2}}dx$.

Solution for the Sample test 1 - Math 205

1. The 50 equal subintervals are determined by $x_i = i$; for $i = 1, 2, ..., 50$. Denoting the right sum by R_{50} we have: $R_{50} = \sum_{i=1}^{50} f(x_i) \cdot \Delta x$, where $\Delta x = \frac{5}{50} = \frac{1}{10}$ and $x_i = \frac{i}{10}$ giving $\mathbf{R}_{50} = \frac{\mathbf{1}}{\mathbf{10}} \sum_{\mathbf{i=1}}^{\mathbf{50}} \left(\mathbf{25} - \frac{\mathbf{i^2}}{\mathbf{100}} \right)$.
   Since f is decreasing on $[0, 5]$, R_{50} is an underestimate.

2. We can write $h(x) = g(u)$ where $g(u) = \int_u^0 \sqrt{1+t^3}dt = -\int_0^u \sqrt{1+t^3}dt$ and $u = x^2$. Therefore: $f(x) \overset{Chain\ Rule}{=} \frac{dg}{du}\frac{du}{dx} = \frac{d}{du}\left(-\int_0^u \sqrt{1+t^3}dt \right) \cdot$
   $(2x) \overset{FTC}{=} -\sqrt{1+u^3} \cdot (2x) = -2x\sqrt{1+(x^2)^3} = -\mathbf{2x}\sqrt{\mathbf{1}+\mathbf{x^6}}$

3. (a) Let $u = \sqrt{x^2+1}; du = xdx / \left(\sqrt{x^2+1}\right)$ and so $I = \int \cos u du = \sin u + C = \sin \sqrt{\mathbf{x^2+1}} + \mathbf{C}$.

   (b) $\int_0^2 \sqrt{x^3+1}x^5 dx = \int_0^2 \left(x^3\sqrt{x^3+1}\right) x^2 dx = \left| \begin{array}{cc} u = x^3+1 \to x^3 = u-1 & x = 0 \to u = 1 \\ du = 3x^2dx \to (1/3)\,du = x^2dx & x = 2 \to u = 9 \end{array} \right| =$
   $\frac{1}{3}\int_1^9 \sqrt{u}(u-1)\,du = \frac{2}{15}(3u-5)\sqrt{u}^3]_1^9 = \frac{1192}{45} = \mathbf{26} + \frac{\mathbf{22}}{\mathbf{45}}$.

4. $\int_1^e \left(\ln^2 x\right) dx = \left| \begin{array}{cc} u & v' \\ \ln^2 x & 1 \\ \frac{2\ln x}{x} & x \end{array} \right| = x\ln^2 x]_1^e - 2\int_1^e \ln x dx = \left| \begin{array}{cc} u & v' \\ \ln x & 1 \\ \frac{1}{x} & x \end{array} \right| =$
   $e - 2\left(x\ln x]_1^e - \int_1^e dx \right) = \mathbf{e - 2}$.

5. Because: $\frac{d}{dx}\ln|\sec x + \tan x| = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$.

6. From $I_n = \int \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$ it follows that:

   (a) $\int \cos^4 x dx = \frac{\sin \mathbf{x} \cos^{\mathbf{3}} \mathbf{x}}{\mathbf{4}} + \frac{\mathbf{3}}{\mathbf{4}}\left(\frac{\sin \mathbf{x} \cos \mathbf{x}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{x} \right) + \mathbf{C}$.

   (b) $\int_0^{\frac{\pi}{2}} \cos^{10} x dx = \frac{9 \times 7 \times 5 \times 3 \times \pi}{10 \times 8 \times 6 \times 4 \times 2 \times 2} = \frac{\mathbf{63\pi}}{\mathbf{512}}$.

7. Using partial fractions we get:

(a) $\int \frac{3xdx}{(x+3)(x-2)}$ with the decomposition:

$\frac{3x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \to 3x = A(x-2) + B(x+3)$

Replace

$x = 2 \to 6 = 5B \to B = \frac{6}{5}$

$x = -3 \to -9 = -5A \to A = \frac{9}{5}$

we get: $\int \frac{3xdx}{(x+3)(x-2)} = \int \left(\frac{9}{5}\frac{1}{x+3} + \frac{6}{5}\frac{1}{x-2} \right) dx = \mathbf{\frac{9}{5}} \ln|\mathbf{x+3}| + \mathbf{\frac{6}{5}} \ln|\mathbf{x-2}| + \mathbf{C}.$

(b) $\int_1^{\sqrt{3}} \frac{x^2}{x^2+1} dx = \left| \begin{array}{c} \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1} \\ \int \frac{x^2}{x^2+1} dx = x - \arctan x + C \end{array} \right| = \int_1^{\sqrt{3}} \frac{x^2}{x^2+1} dx =$
$\boldsymbol{\sqrt{3} - \frac{\pi}{12} - 1}.$

8. $\int \frac{2x - x^3}{\sqrt{4-x^2}} dx$ (since besides the root there are only odd powers of x we get) $= \int \frac{2-x^2}{\sqrt{4-x^2}} x dx =$

$\left| \begin{array}{c} t = \sqrt{4-x^2} \to t^2 = 4 - x^2 \\ 2 - x^2 = t^2 - 2 \\ xdx = -tdt \end{array} \right| = -\int \left(t^2 - 2\right) dt = -\frac{1}{3} t \left(t^2 - 6\right) + C =$
$\boldsymbol{\frac{1}{3}\sqrt{4-x^2}\left(x^2+2\right) + C}.$