Multiple Choice Questions (1-4)

Question 1 *Solve*

$$3^{2x} = 2^{x+1}$$

A) $x = \frac{3\ln(2)}{2\ln(3)}$ **B)** $x = \frac{-\ln(3)}{\ln(2)}$ **C)** $x = \frac{\ln(3)}{\ln(2)-\ln(3)}$ **D)** $x = \frac{\ln(2)}{\ln(3)+2\ln(2)}$ **E)** $x = \frac{\ln(2)}{2\ln(3)-\ln(2)}$

$$\ln(3^{2x}) = \ln(2^{x+1})$$
$$2x\ln(3) = (x+1)\ln(2)$$
$$2x\ln(3) - x\ln(2) = \ln(2)$$
$$x(2\ln(3) - \ln(2)) = \ln(2)$$

$$x = \frac{\ln(2)}{2\ln(3)-\ln(2)}$$

Question 2 *If* $f(x) = \frac{x}{x^2+1}$, *find* $f'(2)$.

A) $\frac{3}{5}$ **B)** $\frac{1}{25}$ $\boxed{\textbf{C)}\ \frac{-3}{25}}$ **D)** $\frac{1}{5}$ **E)** $\frac{-7}{25}$

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$$f'(2) = \frac{-3}{25}$$

Question 3 *Find the equation of the tangent line of the function* $f(x) = \frac{1}{\sqrt{3x+1}}$ *at* $x = 1$.

A) $y = -\frac{3}{16}x - \frac{8}{3}$ B) $y = -\frac{3}{16}x + \frac{11}{16}$ C) $y = \frac{1}{16}x - \frac{3}{16}$ D) $y = \frac{1}{16}x + \frac{7}{16}$

E) $y = \frac{1}{16}x - \frac{2}{3}$

$f(x) = (3x+1)^{-1/2}$

$f'(x) = -\frac{1}{2}(3x+1)^{-3/2}(3)$

$f'(1) = -\frac{1}{2} \cdot \frac{1}{8} \cdot 3 = -\frac{3}{16}$

So $y = -\frac{3}{16}x + b$. The point $(1, \frac{1}{2})$ is on the line. So

$\frac{1}{2} = -\frac{3}{16} + b \quad b = \frac{11}{16}$

Question 4 *Find the following limit.*

$$\lim_{x \to 2} \frac{x-2}{x^3 - 8}$$

A) 12 B) $\frac{1}{3}$ C) $\frac{1}{12}$ D) ∞ E) The limit does NOT exist.

$x^3 - 8 = (x-2)(x^2+2x+4)$

So $\lim_{x \to 2} \frac{x-2}{x^3-8} = \lim_{x \to 2} \frac{x-2}{(x-2)(x^2+2x+4)}$

$= \lim_{x \to 2} \frac{1}{x^2+2x+4} = \frac{1}{12}$

Long Answer Questions (5-7)

Question 5 (14 points)

Using only the definition of derivative as a limit, calculate $f'(x)$ where

$$f(x) = \frac{2}{3-x}.$$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{2}{3-x-h} - \frac{2}{3-x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2(3-x)}{(3-x-h)(3-x)} - \frac{2(3-x-h)}{(3-x-h)(3-x)}}{h}$$

$$= \lim_{h\to 0} \frac{6-2x-(6-2x-2h)}{h(3-x-h)(3-x)} = \lim_{h\to 0} \frac{2h}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{2}{(3-x-h)(3-x)} = \frac{2}{(3-x)^2}$$

<u>Check:</u> $f(x) = 2(3-x)^{-1}$

$$f'(x) = -2(3-x)^{-2}(-1) = \frac{2}{(3-x)^2}$$

Question 6 (12 points) For the following, you do not need to simplify your answers.

- *Suppose 1,000 dollars is invested at a rate of 4 percent, compounded 6 times per year. How much money is in the account after 3 years?*
- *Suppose 1,000 dollars is invested at a rate of 4 percent, compounded 6 times per year. How long does it take for the money to triple?*
- *Suppose 2,000 dollars is invested in an account that compounds continuously. The interest is unknown, but you do know that the money tripled in 21 years. What must the interest rate have been?*

Solution a) $P(t) = P_0\left(1+\frac{r}{n}\right)^{nt} = 1{,}000\left(1+\frac{.04}{6}\right)^{6t}$

$P(3) = 1{,}000\,(1.00666\ldots)^{18}$

b) $P(t) = 1{,}000\,(1.00666\ldots)^{6t}$

Let $3{,}000 = 1{,}000(1.00666\ldots)^{6t}$. Solve for t

$3 = (1.00666\ldots)^{6t}$

$\ln(3) = 6t\ln(1.00666\ldots)$

$$t = \frac{\ln(3)}{6\ln(1.00666\ldots)}$$

c) $P(t) = 2{,}000\,e^{rt}$. We know $P(21) = 6{,}000$.

So $6{,}000 = 2{,}000\,e^{r(21)}$

$3 = e^{21r}$

$\ln(3) = 21r$

$$r = \frac{\ln(3)}{21}$$

Question 7 (10 points)

The function $y = f(x)$ is defined implicitly by

$$\sqrt{2x+3y} = y+2$$

Find the equation of the tangent line to the graph determined by the above equation at $(3,1)$.

Solution

$$(2x+3y)^{1/2} = y+2$$

$$\frac{1}{2}(2x+3y)^{-1/2}\left(2+3\frac{dy}{dx}\right) = \frac{dy}{dx}$$

Plug in $(3,1)$

$$\frac{1}{2}(9)^{-1/2}\left(2+3\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{1}{6}\left(2+3\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{1}{3}+\frac{1}{2}\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{1}{3} = \frac{1}{2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2/3$$

So $y = \frac{2}{3}x + b$. Plug in $(3,1)$

$$1 = \frac{2}{3}(3) + b \qquad b = -1$$

$$\boxed{y = \frac{2}{3}x - 1}$$

Multiple Choice Questions (1-4)

Question 1 *Solve*

$$2^{3x} = 5^{x+1}$$

A) $x = \frac{\ln(5)}{3\ln(2)-\ln(5)}$ **B)** $x = \frac{-\ln(2)}{\ln(5)}$ **C)** $x = \frac{\ln(2)}{\ln(2)-\ln(5)}$ **D)** $x = \frac{\ln(2)}{\ln(5)+3\ln(2)}$

E) $x = \frac{\ln(2)}{2\ln(5)+\ln(2)}$

$2^{3x} = 5^{x+1}$

$\ln(2^{3x}) = \ln(5^{x+1})$

$3x\ln(2) = (x+1)(\ln(5))$

$3x\ln(2) - x\ln(5) = \ln(5)$

$x(3\ln(2) - \ln(5)) = \ln(5)$

$x = \dfrac{\ln(5)}{3\ln(2)-\ln(5)}$

Question 2 *If* $f(x) = \frac{x^2}{2x+1}$, *find* $f'(3)$.

A) $\frac{3}{7}$ **B)** $\frac{24}{49}$ **C)** $\frac{-3}{49}$ **D)** $\frac{1}{49}$ **E)** $\frac{-7}{49}$

$f'(x) = \dfrac{(2x+1)(2x) - x^2(2)}{(2x+1)^2}$ by quotient rule

$f'(3) = \dfrac{7(6) - 9(2)}{49} = \dfrac{24}{49}$

Question 3 *Find the equation of the tangent line of the function* $f(x) = \frac{1}{\sqrt{3x+3}}$ *at* $x = 2$.

A) $y = -\frac{3}{18}x - \frac{8}{3}$ B) $y = -\frac{1}{18}x + \frac{11}{18}$ C) $y = \frac{1}{18}x - \frac{3}{9}$ $\boxed{\text{D) } y = -\frac{1}{18}x + \frac{4}{9}}$

E) $y = \frac{1}{18}x - \frac{2}{3}$

$f(x) = (3x+3)^{-1/2}$

$f'(x) = -\frac{1}{2}(3x+3)^{-3/2}(3)$

$f'(2) = -\frac{1}{2}(9)^{-3/2} \cdot 3 = -\frac{1}{2} \cdot \frac{3}{27} = -\frac{1}{18}$

So $y = -\frac{1}{18}x + b$, The point $(2, \frac{1}{3})$ is on the line. So $\frac{1}{3} = -\frac{1}{9} + b$

So $b = \frac{4}{9}$

Question 4 *Find the following limit.*

$$\lim_{x \to 3} \frac{x-3}{x^3 - 27}$$

A) 27 B) $\frac{1}{27}$ C) $\frac{1}{3}$ D) ∞ E) The limit does NOT exist.

$x^3 - 27 = x-3(x^2+3x+9)$ Use long or synthetic division

So $\lim_{x \to 3} \frac{x-3}{x^3-27} = \lim \frac{x-3}{(x-3)(x^2+3x+9)} = \lim_{x \to 3} \frac{1}{x^2+3x+9} = \frac{1}{27}$

Long Answer Questions (5-7)

Question 5 (14 points)
Using only the definition of derivative as a limit, calculate $f'(x)$ where

$$f(x) = \frac{3}{4-x}.$$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{4-x-h} - \frac{3}{4-x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3(4-x)}{(4-x-h)(4-x)} - \frac{3(4-x-h)}{(4-x-h)(4-x)}}{h}$$

$$= \lim_{h\to 0} \frac{(12-3x) - (12-3x-3h)}{h(4-x-h)(4-x)} = \lim_{h\to 0} \frac{3h}{h(4-x-h)(4-x)}$$

$$= \lim_{h\to 0} \frac{3}{(4-x-h)(4-x)} = \frac{3}{(4-x)^2}$$

Check it: $f(x) = 3(4-x)^{-1}$

$$f'(x) = -3(4-x)^{-2}(-1)$$

$$= \frac{3}{(4-x)^2} \checkmark$$

Question 6 (12 points) For the following, you do not need to simplify your answers.

- *Suppose 1,000 dollars is invested at a rate of 3 percent, compounded 4 times per year. How much money is in the account after 5 years?*
- *Suppose 1,000 dollars is invested at a rate of 4 percent, compounded 3 times per year. How long does it take for the money to double?*
- *Suppose 3,000 dollars is invested in an account that compounds continuously. The interest is unknown, but you do know that the money tripled in 18 years. What must the interest rate have been?*

Solution

a) $P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$

$= 1{,}000\left(1+\frac{.03}{4}\right)^{4t}$

$P(5) = 1{,}000\,(1.0075)^{20}$

b) $P(t) = 1{,}000\left(1+\frac{.04}{3}\right)^{3t}$

$= 1{,}000\,(1.0133\ldots)^{3t}$

Solve $2{,}000 = 1{,}000\,(1.0133\ldots)^{3t}$

$2 = (1.0133\ldots)^{3t}$

$\ln(2) = 3t\ln(1.0133\ldots)$

$t = \dfrac{\ln(2)}{3\ln(1.0133\ldots)}$

c) $P(t) = 3{,}000\,e^{rt}$. We know $P(18) = 9{,}000$

So $9{,}000 = 3{,}000\,e^{18r}$

$3 = e^{18r}$

$\ln(3) = 18r \Rightarrow r = \dfrac{\ln(3)}{18}$

Question 7 (10 points)

The function $y = f(x)$ is defined implicitly by

$$\sqrt{3x + 2y} = y - 1$$

Find the equation of the tangent line to the graph determined by the above equation at $(2, 5)$.

Solution

$$(3x+2y)^{1/2} = y-1$$

$$\frac{1}{2}(3x+2y)^{-1/2}\left(3+2\frac{dy}{dx}\right) = \frac{dy}{dx} \quad \text{Plug in } (2,5)$$

$$\frac{1}{2}(16)^{-1/2}\left(3+2\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{1}{8}\left(3+2\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{3}{8} + \frac{1}{4}\frac{dy}{dx} = \frac{dy}{dx}$$

$$3/8 = \frac{3}{4}\,dy/dx \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

So $y = \frac{1}{2}x + b$. Plug in $(2,5)$

$$5 = 2\left(\tfrac{1}{2}\right) + b$$

$$b = 4$$

$$\boxed{y = \frac{1}{2}x + 4}$$