MATH. 1339 B

Winter 2013
Midterm 1.
Instructor: Maryam Hosseini

Family Name:____________________ First Name:____________________

Student Number:____________________

Instructions:

- You have 80 minutes to complete this exam.
- This is a closed book exam and no notes of any kind are allowed. **Don't use your own scrap paper! Use the last page or the backs of pages for rough work.**
- Only use of approved calculators is permitted.
- All of the questions require a complete solution and are worth 20 points. Use backs of the pages if necessary.
- Good Luck! Bonne chance!

Question	1	2	3	4	5
Score					

Question 1. Evaluate the limits:

a) $\lim_{x\to 3} \frac{x-3}{\sqrt{3x-6}-\sqrt{2x-3}}$

b) $\lim_{x\to 0} \frac{x}{|x|}$

Solution: a) $\lim_{x\to 3} \frac{x-3}{\sqrt{3x-6}-\sqrt{2x-3}} = \frac{0}{0}$

$= \lim_{x\to 3} \frac{(x-3)(\sqrt{3x-6}+\sqrt{2x-3})}{(3x-6)-(2x-3)}$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(\sqrt{3x-6}+\sqrt{2x-3})}{\cancel{(x-3)}}$

$= \lim_{x\to 3} \frac{\sqrt{3x-6}+\sqrt{2x-3}}{1} = \sqrt{9-6}+\sqrt{6-3} = \sqrt{3}+\sqrt{3} = 2\sqrt{3}.$

b) $\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$

$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0^-} (-1) = -1$

$\Rightarrow \lim_{x\to 0} \frac{x}{|x|}$ does not exist.

Question 2. Find the values of a and b such that the function f is continuous at any point of $\mathbb{R}$.

$$f(x) = \begin{cases} ax+b, & x \leq 0; \\ x^2+a, & 0 < x \leq 1; \\ bx+a, & x > 1. \end{cases}$$

Solution :

to have continuity at $x=0$: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

$$\Rightarrow \lim_{x\to 0^-} ax+b = b = \lim_{x\to 0^+} x^2+a = a$$

$$\Rightarrow \boxed{a=b}\ (1)$$

to have continuity at $x=1$: $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$

$$\Rightarrow \lim_{x\to 1^-} x^2+a = 1+a = \lim_{x\to 1^+} bx+a = b+a$$

$$\Rightarrow 1+\cancel{a} = b+\cancel{a} \Rightarrow \boxed{b=1}\ (2)$$

$$(2),(1) \Rightarrow \boxed{a=b=1.}$$

Question 3. let $f(x) = -3x^2 - x + 4$.

a) Using the "first principle of derivative" find $f'(x)$.

b) Determine the equation of tangent line to the curve of f at the point $x = -1$.

Solution :

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-3(x+h)^2 - (x+h) + 4 + 3x^2 + x - 4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-3x^2} - 6xh - 3h^2 \cancel{-x} - h \cancel{+4} + \cancel{3x^2} \cancel{+x} \cancel{-4}}{h}$$

$$= \lim_{h\to 0} \frac{-6xh - 3h^2 - h}{h} = \lim_{h\to 0} \frac{\cancel{h}(-6x - 3h - 1)}{\cancel{h}} = -6x - 1$$

Equation of tangent line at $x = -1$:

$$y - f(-1) = f'(1)\left(x - (-1)\right)$$

$$y - 2 = 5(x + 1)$$

$$\boxed{y = 5x + 7}$$

Question 4. Differentiate each function:

a) $f(x) = 2(\sqrt{x}+1)(x^3-2x^2+3)^2$

b) $g(x) = \dfrac{\sqrt{x^2-x}}{x+1}$

Solution: a) $f'(x) = 2(\sqrt{x}+1)'(x^3-2x^2+3)^2 + 2(\sqrt{x}+1)\left((x^3-2x^2+3)^2\right)'$

$$f'(x) = 2\left(\frac{1}{2\sqrt{x}}+0\right)(x^3-2x^2+3)^2 + 2(\sqrt{x}+1)\left(2(x^3-2x^2+3)(3x^2-4x+0)\right)$$

$$= \frac{1}{\sqrt{x}}(x^3-2x^2+3)^2 + 2(\sqrt{x}+1)2(x^3-2x^2+3)(3x^2-4x)$$

$$= \frac{1}{\sqrt{x}}(x^3-2x^2+3)^2 + 4(\sqrt{x}+1)(3x^2-4x)(x^3-2x^2+3)$$

$$= (x^3-2x^2+3)\left[\frac{1}{\sqrt{x}}(x^3-2x^2+3) + 4(\sqrt{x}+1)(3x^2-4x)\right].$$

b) $$g'(x) = \frac{(\sqrt{x^2-x})'(x+1) - (x+1)'(\sqrt{x^2-x})}{(x+1)^2} = \frac{\left((x^2-x)^{\frac{1}{2}}\right)'(x+1) - (1+0)\sqrt{x^2-x}}{(x+1)^2}$$

$$= \frac{\frac{1}{2}(x^2-x)^{\frac{1}{2}-1}(2x-1)(x+1) - \sqrt{x^2-x}}{(x+1)^2} = \frac{\frac{(2x-1)(x+1)}{2\sqrt{x^2-x}} - \sqrt{x^2-x}}{(x+1)^2}$$

$$= \frac{(2x-1)(x+1) - 2(x^2-x)}{2\sqrt{x^2-x}\,(x+1)^2}.$$

Question 5. Let $f(x) = x^3 - 6x^2 + 9x$.
a) Find the domain of f.
b) Find the set of critical points, local extremes and the intervals of increase and decrease.
c) Sketch the graph of the function.

Solution :

a) $\mathrm{Dom}(f) = \mathbb{R}$ since f is a polynomial.

b) $f'(x) = 3x^2 - 12x + 9$ always exists

$\Rightarrow$ critical points $= \{x : f'(x) = 0\}$

$f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0 \Rightarrow x_1, x_2 = \dfrac{6 \pm \sqrt{36-27}}{3}$

$= \dfrac{6 \pm 3}{3} = \begin{cases} 3 \\ 1 \end{cases}$

$\Rightarrow$ critical points $= \{1, 3\}$

x	$-\infty$		1		3		$+\infty$
f'		+	0	−	0	+	
f	$-\infty$	↗	4	↘	0	↗	$+\infty$
			local max		local min		

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^3 = -\infty$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^3 = +\infty$

interval of increase : $(-\infty, 1) \cup (3, +\infty)$

interval of decrease : $(1, 3)$

Question 5 (suite)