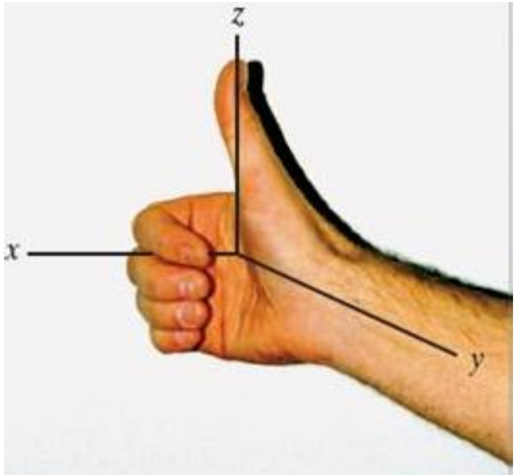
Lecture 3

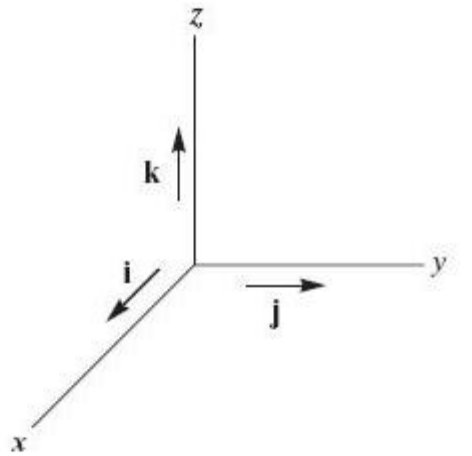
ECOR1101 Mechanics I

Cartesian Vectors in 3D

- 3D vectors are best represented in Cartesian vector notation
- 3D Coordinate System:
  - Right-handed CS
  - Thumb $\rightarrow$ +ve z-axis
  - Fingers curled around z-axis, sweeps from x-axis to y-axis
- Cartesian Unit Vectors
  - Cartesian unit vectors **i, j, k** designate vectors in the x, y, z directions respectively.
  - The positive directions of the unit vectors are as shown in fig.



$$\mathbf{u} = \frac{\mathbf{U}}{|U|}$$



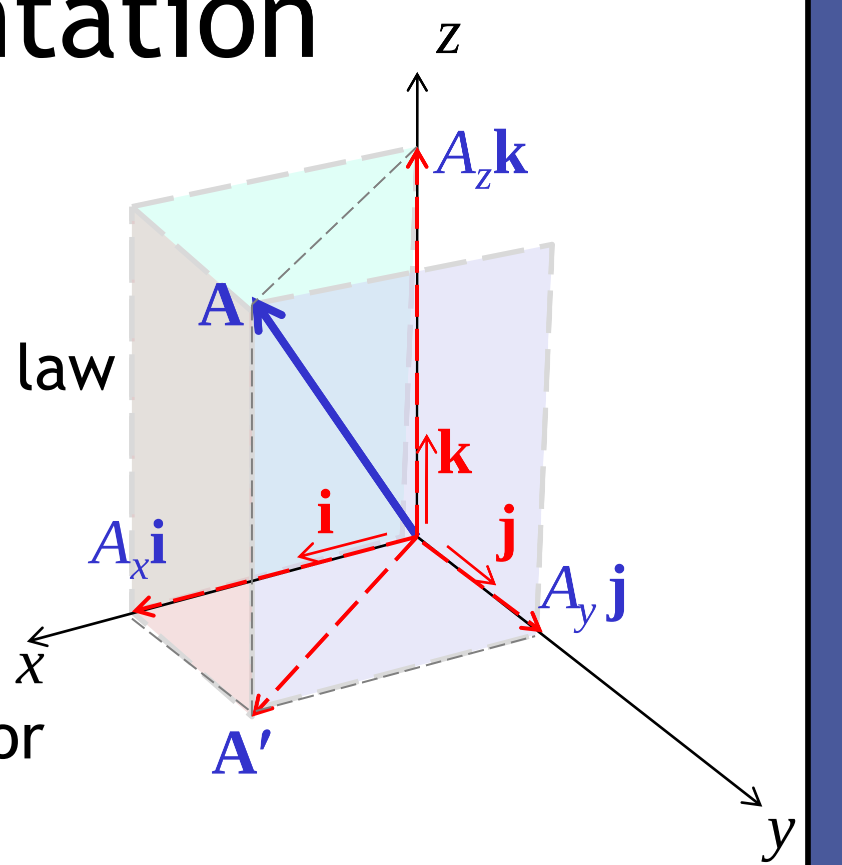
Cartesian Vector Representation

- Resolution of **A** into the Cartesian unit vectors will require two successive application of parallelogram law

$$\mathbf{A} = \mathbf{A}' + A_z\mathbf{k}$$

$$\mathbf{A}' = A_x\mathbf{i} + A_y\mathbf{j}$$

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$



- Magnitude of Cartesian vector

$$A = \sqrt{A'^2 + A_z^2}$$

$$A' = \sqrt{A_x^2 + A_y^2}$$

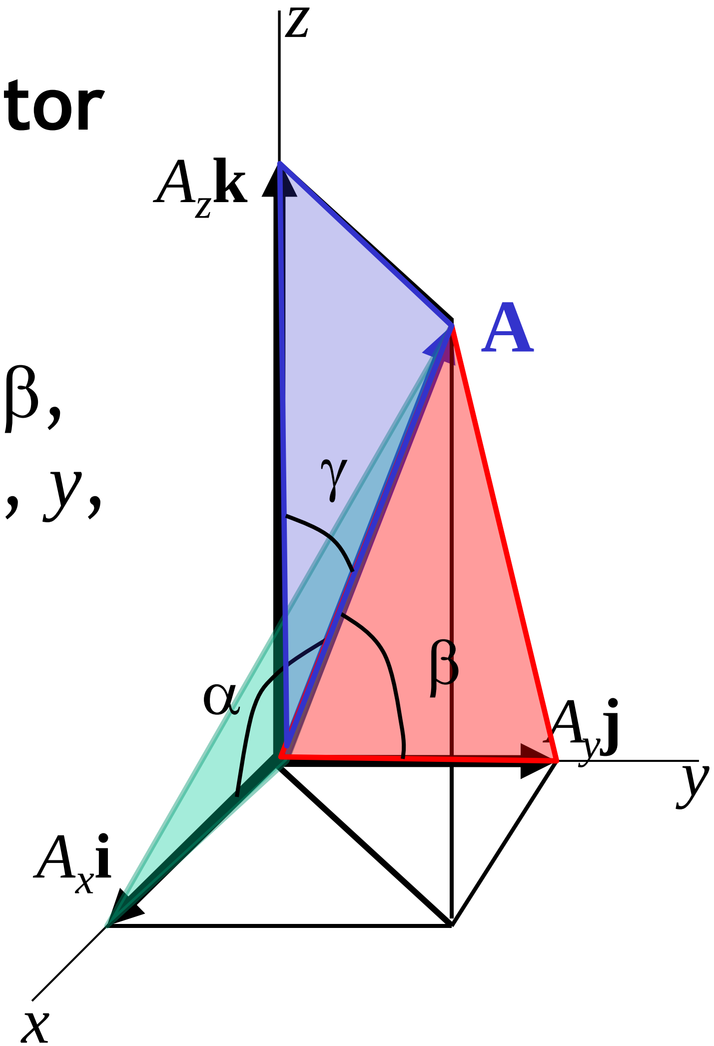
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Cartesian Vector Representation

Direction of Cartesian Vector

- The direction of the Cartesian vector **A** is defined by the angles α, β, and γ it makes with the *x*, *y*, and *z* axes, respectively.

$$\left.\begin{aligned}\cos\alpha &= \frac{A_x}{A}\\ \cos\beta &= \frac{A_y}{A}\\ \cos\gamma &= \frac{A_z}{A}\end{aligned}\right\}\textbf{Direction cosines of A}$$



Cartesian Vector Representation

- A vector **A** can be represented using unit vectors as:
  $\mathbf{A} = A\mathbf{u}_A$, where $\mathbf{u}_A$ is a unit vector in the direction of **A**

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

- Since the magnitude of a unit vector is 1;

$$u_A = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

  - Therefore if any two coordinate angles (direction cosines) are known we can find the third.

Cartesian Vector Representation

- The direction of **A** can be found from two angles, ϕ and θ (altitude and azimuth)

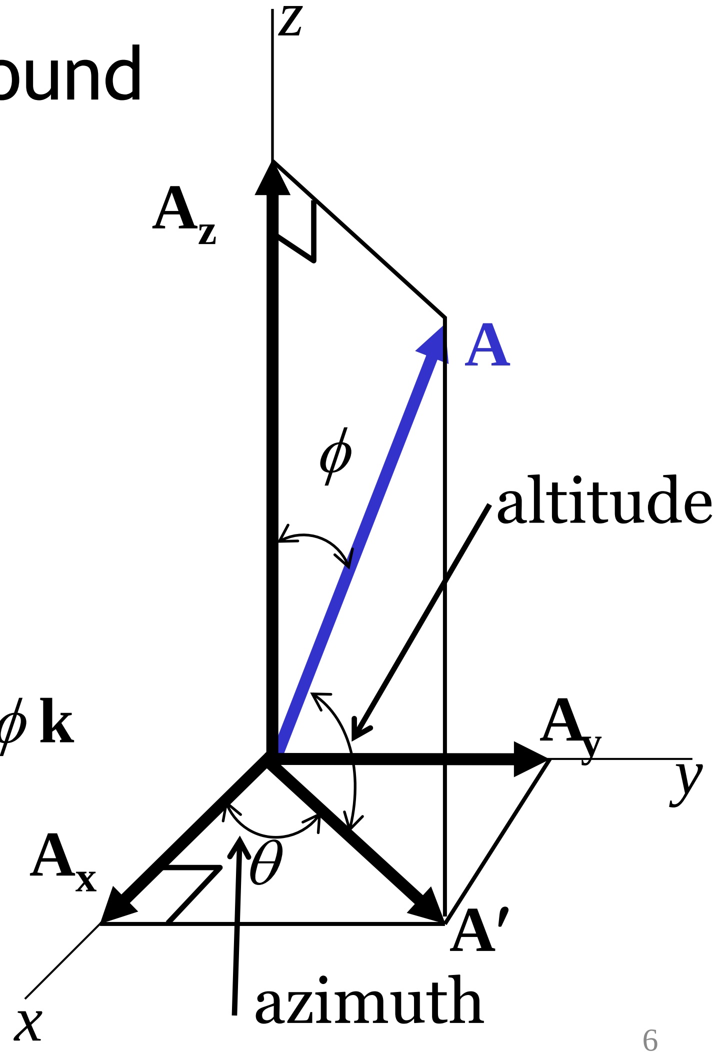
$$A_z = A\cos\phi$$

$$A' = A\sin\phi$$

$$A_x = A'\cos\theta = A\sin\phi\cos\theta$$

$$A_y = A'\sin\theta = A\sin\phi\sin\theta$$

$$\mathbf{A} = A\sin\phi\cos\theta\,\mathbf{i} + A\sin\phi\sin\theta\,\mathbf{j} + A\cos\phi\,\mathbf{k}$$



Addition of Cartesian Vectors

- Given two vectors **A** and **B**;

  $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

  $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$

- We can add **A** and **B** using Cartesian components as follows:

  $\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$

- We can also subtract A and B using Cartesian components as follows:

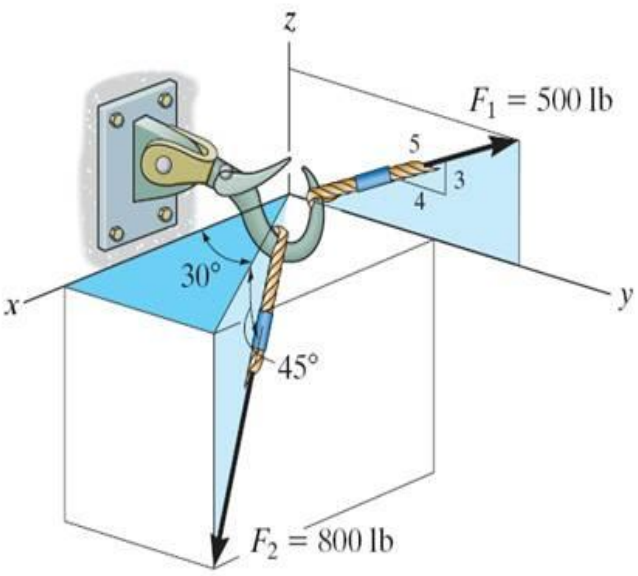
  $\mathbf{R} = \mathbf{A} - \mathbf{B} = (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k}$

- General formulation

  $\mathbf{R} = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$

Sample Problem

F2-18 Determine the resultant force acting on the hook.



<u>Solution Procedure</u>

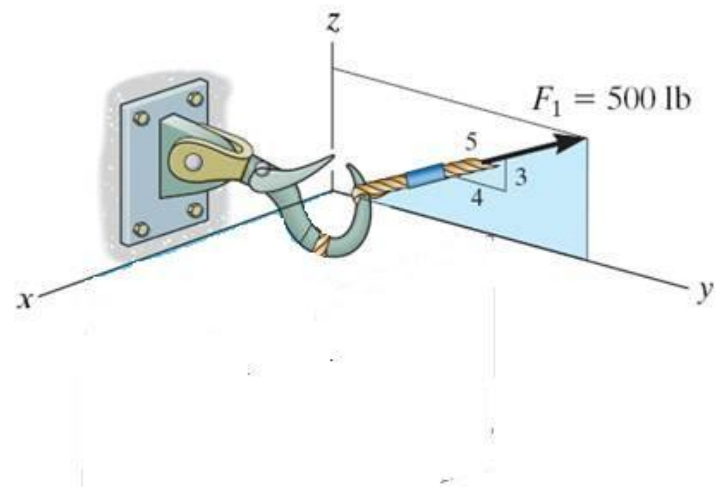
1) Using geometry and trigonometry, write $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ in Cartesian vector form.
2) Then add the two forces (by adding *x* and *y* components of the forces).

Resolve force $\mathbf{F_1}$.

$F_{1x} = 0 = 0$ lb

$F_{1y} = 500\ (4/5) = 400$ lb

$F_{1z} = 500\ (3/5) = 300$ lb



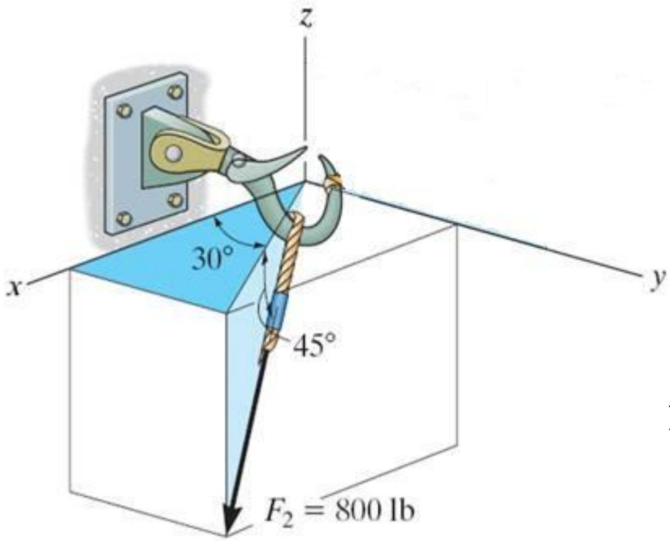
Write $\mathbf{F_1}$ in Cartesian vector form (don't forget the units!).

$\mathbf{F_1} = \{0\mathbf{i} + 400\mathbf{j} + 300\mathbf{k}\}$ lb

Resolve force $\mathbf{F_2}$.

We are given only two angles, α and γ.

So we need to resolve $\mathbf{F_2}$ into the *z*-axis and the *xy*-plane.



$$F_{2z} = F_2 \sin 45^o = 800 \times \sin 45^o = 565.69 \text{ lb.}$$

$$F_{2xy} = F_2 \cos 45^o = 800 \times \cos 45^o = 565.69 \text{ lb.}$$

$$F_{2x} = F_{2xy} \cos 30^o = 565.69 \times \cos 30^o = 489.90 \text{ lb.}$$

$$F_{2y} = F_{2xy} \sin 30^o = 569.69 \times \sin 30^o = 282.84 \text{ lb.}$$

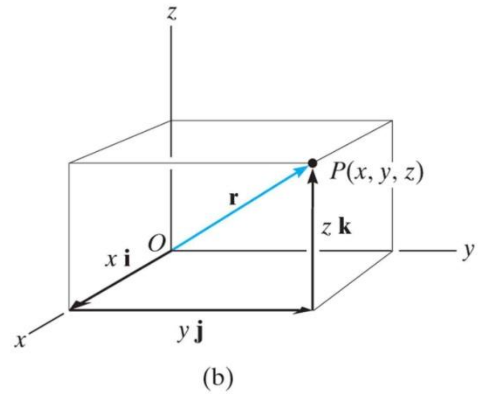
$$\mathbf{F_2} = \{489.90\,\mathbf{i} + 282.84\,\mathbf{j} - 565.69\,\mathbf{k}\} \text{ lb}$$

$$\mathbf{F_R} = \mathbf{F_1} + \mathbf{F_2} = \{(489.90)\mathbf{i} + (400 + 282.84)\mathbf{j} + (300 - 565.69)\mathbf{k}\} \text{ lb}$$

$$\mathbf{F_R} = \{490\,\mathbf{i} + 683\,\mathbf{j} - 266\,\mathbf{k}\} \text{ lb.}$$

Position Vectors

- A position vector $\mathbf{r}$ is a fixed vector which defines a point in space relative to another point e.g. a point P relative to the origin O



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- $\mathrm{O}(x = 0, y = 0, z = 0)$ and $\mathrm{P}(x, y, z)$

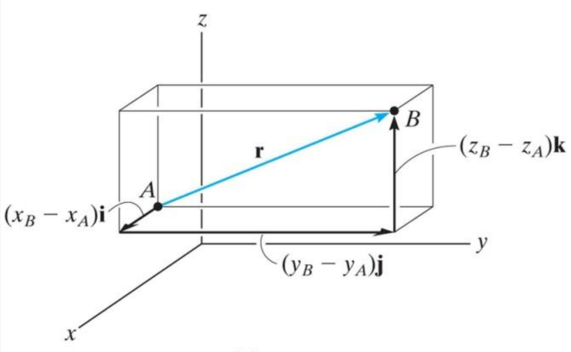
  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- In general if a position vector is directed from point $A(x_A, y_A, z_A)$ to $B(x_B, y_B, z_B)$, then

  $\mathbf{r}_{\mathbf{AB}} = \mathbf{r}_{\mathbf{B}} - \mathbf{r}_{\mathbf{A}}$

  $= (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$

  $= (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$



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Position Vectors

- The magnitude of the position vector $\mathbf{r_{AB}}$ is given as:

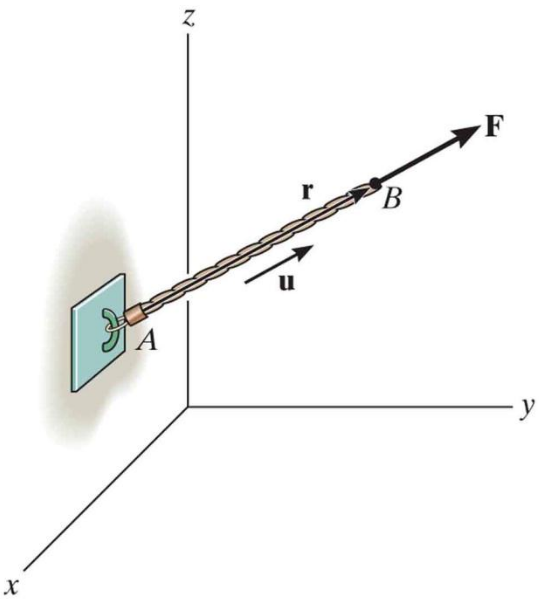
$$r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

- The direction of the position vector $\mathbf{r_{AB}}$ is given by the direction cosines of unit vector of $\mathbf{r_{AB}}$; $\cos\alpha$, $\cos\beta$, and $\cos\gamma$

$$\mathbf{u} = \frac{\mathbf{r_{AB}}}{r_{AB}} = \cos\alpha\ \mathbf{i} + \cos\beta\ \mathbf{j} + \cos\gamma\ \mathbf{k}$$

Force Vector Along a Line

- In 3-dimensional statics, the force **F** can be specified by two points, A and B through which passes the line of action of **F**
- **F** can also be represented by the position vector, **r** from point A to B.
- Hence:



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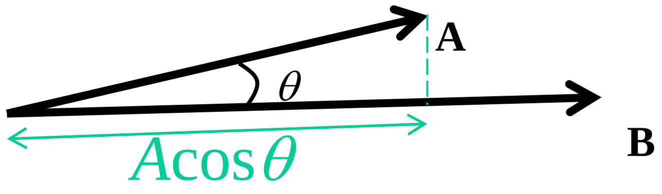
$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\,\mathbf{i} + (y_B - y_A)\,\mathbf{j} + (z_B - z_A)\,\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

Dot Product (Scalar Product)

- In statics it is often important to find:
  - the angle between two forces (vectors).
  - the perpendicular and parallel components of a force (vector) to a line.
- The dot product is required to solve these problems.
- The dot product of 2 vectors is the product of the magnitudes of the vectors and the cosine of the angle between the vectors



$$\mathrm{A} \cdot \mathrm{B} = AB \cos \theta$$

$$0^\circ \leq \theta \leq 180^\circ$$

Dot Product

- Laws of Operation
  - Commutative law

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}\ (AB\cos\theta = BA\cos\theta)$$

  - Multiplication by scalar

$$a(\mathbf{A}\cdot\mathbf{B}) = (a\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(a\mathbf{B})$$

  - Distributive law

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})+(\mathbf{A}\cdot\mathbf{C})$$

Dot Product

Cartesian vector formulation

- Dot product of two collinear unit vectors is 1

$\mathbf{i}\cdot\mathbf{i} = (1)(1)\cos(0°) = 1$

- Dot product of two perpendicular vectors is 0

$\mathbf{i}\cdot\mathbf{j} = (1)(1)\cos(90°) = 0$

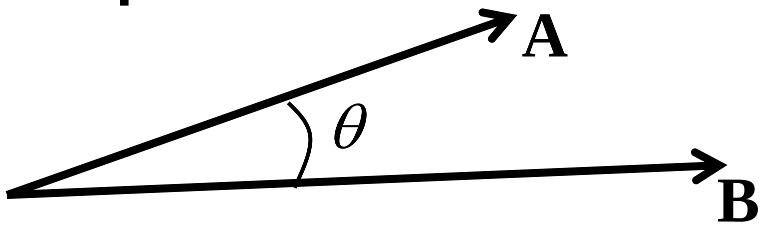
$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_xB_x(\mathbf{i}\cdot\mathbf{i}) + A_xB_y(\mathbf{i}\cdot\mathbf{j}) + A_xB_z(\mathbf{i}\cdot\mathbf{k}) \\
&\quad + A_yB_x(\mathbf{j}\cdot\mathbf{i}) + A_yB_y(\mathbf{j}\cdot\mathbf{j}) + A_yB_z(\mathbf{j}\cdot\mathbf{k}) \\
&\quad + A_zB_x(\mathbf{k}\cdot\mathbf{i}) + A_zB_y(\mathbf{k}\cdot\mathbf{j}) + A_zB_z(\mathbf{k}\cdot\mathbf{k})
\end{aligned}
$$

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + AyB_y + A_zB_z$$

Dot Product

- The angle formed by two intersecting vectors can be determined by dot product as:

$$\theta = \mathbf{cos}^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$



$$0^\circ \le \theta \le 180^\circ$$

- The dot product can be found from:

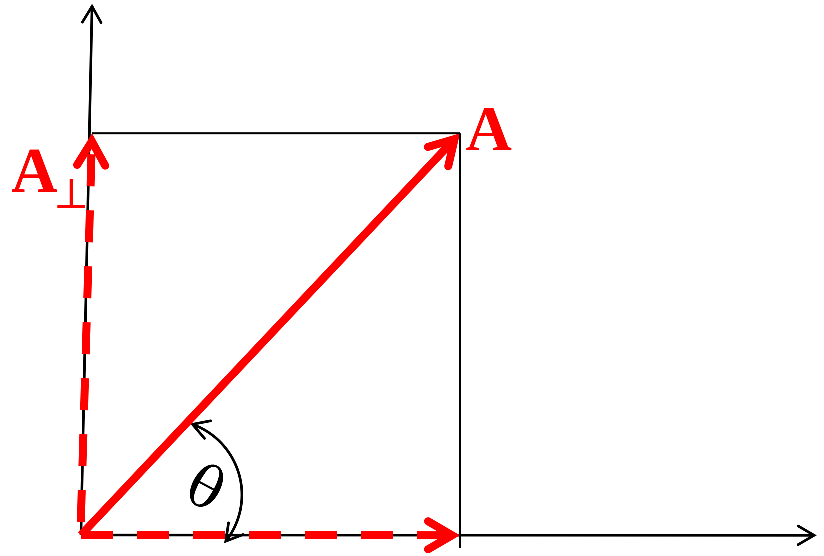
$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Dot Product

- The components of a force parallel and perpendicular to a given line can be determined with the dot product.
- The magnitude of the component of **A** parallel to *x*-axis is given by the dot product of **A** and $\mathbf{u_x}$ (unit vector along *x*-*axis*)

$$A_x = A\cos\theta = \mathbf{A}\cdot\mathbf{u_x}$$

$$\mathbf{A_x} = A_x\mathbf{u_x}$$



$\mathbf{A_x} = A\cos\theta\,\mathbf{u_x}$

The component of the force **A** perpendicular to the *x*-axis can be written as:

$$\mathbf{A} = \mathbf{A_x} + \mathbf{A_\perp}$$

$$\mathbf{A_\perp} = \mathbf{A} - \mathbf{A_x}$$

$A_\perp = A\sin\theta$ or $A_\perp = \sqrt{A^2 - A_x^2}$

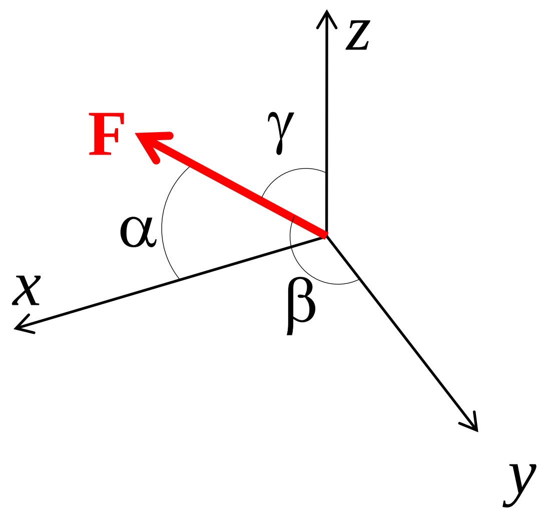
Sample Problem

Find the direction cosines between the force **F** = (4**i** + 6**j** + 7**k**) N and each of the three axes, *x*, *y*, *z*.

$$\mathbf{F} = F\,\mathbf{u}_F$$

$$\mathbf{u}_F = \frac{\mathbf{F}}{F} = \frac{4\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}}{\sqrt{4^2 + 6^2 + 7^2}}$$

$$\mathbf{u}_F = 0.398\,\mathbf{i} + 0.597\,\mathbf{j} + 0.6965\,\mathbf{k}$$



$$\cos\alpha = \frac{\mathbf{u}_F \cdot \mathbf{i}}{1 \times 1} = 0.398 \Rightarrow \alpha = 66.5^o$$

$$\cos\beta = \frac{\mathbf{u}_F \cdot \mathbf{j}}{1 \times 1} = 0.597 \Rightarrow \beta = 53.3^o$$

$$\cos\gamma = \frac{\mathbf{u}_F \cdot \mathbf{k}}{1 \times 1} = 0.6965 \Rightarrow \gamma = 45.9^o$$