Notes

Math 154 Midterm #1 Review

1. Indicate whether the statement is True (T) or False(F).

#	Statement	T	F
1	The function $f(x) = x\cos(x)$ is an even function.		F
2	The function $f(x) = \frac{x^2-1}{(2x-1)(x+1)}$ has exactly one vertical asymptote.	T	
3	Radioactive decay can be modeled using the formula $W(t) = W_0 e^{-\lambda t}$, $\lambda > 0$	T	
4	If $\lim_{x \to 3} f(x)g(x)$ exists, then the limit must be $f(3)g(3)$.		F
5	The graph of an exponential function on a log-log plot forms a straight line.		F

2. a) Find the function $(G \circ G)(x)$ together with its domain if $G(x) = \frac{1-x}{1+x}$.

Let $u = G(x)$ then $G(u) = \frac{1-u}{1+u} \Rightarrow G(G(x)) = \frac{1-G(x)}{1+G(x)} =$

$$= \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{1+x-(1-x)}{1+\cancel{x}+1-\cancel{x}} = \frac{\cancel{1}+x-\cancel{1}+x}{2} = \frac{2x}{2} = x$$

Function	Domain	Range
Inner G(x)	$(-\infty,-1)\cup(-1,\infty)$	$(-\infty,-1)\cup(-1,\infty)$
Outer G(x)	$(-\infty,-1)\cup(-1,\infty)$	$(-\infty,-1)\cup(-1,\infty)$
GoG	$\boxed{(-\infty,-1)\cup(-1,\infty)}$	

Note that range of G is equal to the domain of G.

b) Given that $F(x) = \cos^2(x+7)$, find f(x), g(x) and h(x) such that $F(x) = (f \circ g \circ h)(x)$.

Take $h(x) = x+7$, $g(x) = \cos(x)$, $f(x) = x^2$

then $g(h(x)) = \cos(x+7)$

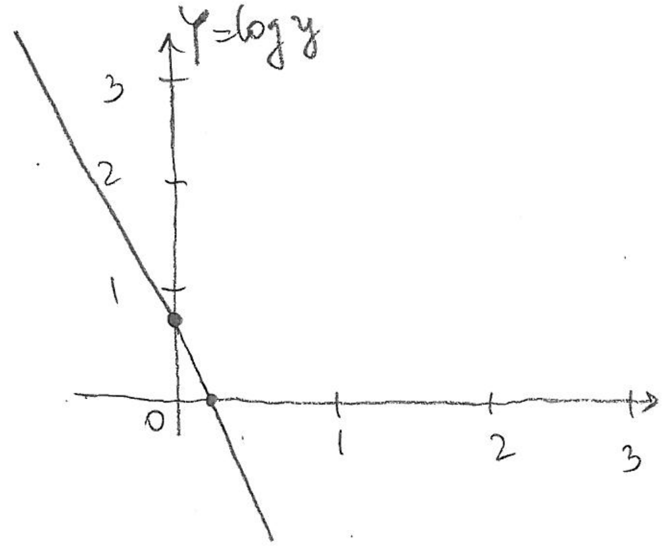
$$f(g(h(x))) = [\cos(x+7)]^2 = \cos^2(x+7)$$

3. Transform the function $y = 5 \cdot x^{-4}$ into a linear one. Sketch the graph and indicate the intercepts.

$$\log(y) = \log(5 \cdot x^{-4})$$

$$\underbrace{\log(y)}_{Y} = \log 5 - 4 \underbrace{\log x}_{X} \quad \text{or} \quad Y = \underbrace{\log 5}_{\approx .7} - 4X$$

log-log plot



$$Y=0 \Rightarrow 4X = \log 5 \Rightarrow X = \frac{\log 5}{4} \approx 0.2$$

4. Find the following limits, if they exist. If a limit does not exist, explain why.

a) $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases} \ln(-x) & \text{if } x<-1 \\ x^2 & \text{if } -1\le x\le 1 \\ \dfrac{3-2x}{x^3} & \text{if } x>1 \end{cases}$

At $\underline{x=-1}$:

(LL) $\lim_{x\to -1^-} \ln(-x) = 0$

(RL) $\lim_{x\to -1^+} x^2 = (-1)^2 = 1$

LL ≠ RL

$\lim_{x\to -1} f(x)$ $\boxed{\text{DNE}}$

At $\underline{x=1}$:

(LL) $\lim_{x\to 1^-} x^2 = 1^2 = 1$

(RL) $\lim_{x\to 1^+} \frac{3-2x}{x^3} = \frac{3-2}{1} = 1$

LL = RL = 1

$\lim_{x\to 1} f(x) =$

b) $\lim_{x\to -4} \frac{x^2-3x-4}{x^2-16} = \lim_{x\to -4} \frac{\cancel{(x-4)}(x+1)}{\cancel{(x-4)}(x+4)} = \lim_{x\to -4} \frac{x+1}{x+4}$

(LL) $\lim_{x\to -4^-} \frac{x+1}{x+4} = \infty$

-4.1

(RL) $\lim_{x\to -4^+} \frac{x+1}{x+4} = -\infty$

-3.9

$\lim_{x\to -4} f(x)$ $\boxed{\text{DNE}}$

$x=-4$

! ** $\underline{\text{Note}}$ only the one-sided limits $\underline{\text{around}}$ $\underline{\text{vertical asymptotes}}$ are to be evaluated numerically. Marks will be lost if other limits are evaluated numerically ** !

$\lim_{x\to 0} x^2 = 0$ cannot use Quotient Law

c) $\lim_{x\to 0} \frac{2-\sqrt{4-x^2}}{x^2} \cdot \frac{2+\sqrt{4-x^2}}{2+\sqrt{4-x^2}} = \lim_{x\to 0} \frac{\overbrace{4-(4-x^2)}^{=\cancel{4}-\cancel{4}+x^2}}{x^2(2+\sqrt{4-x^2})}$

$= \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}(2+\sqrt{4-x^2})} = \frac{1}{2+\sqrt{4}} = \boxed{\frac{1}{4}}$

d) $\lim_{x\to 3} \frac{x^2-9}{|3-x|}$ $\quad f(x) = \begin{cases} \frac{(x-3)(x+3)}{3-x} = -(x+3) & \text{if } 3-x \geq 0, x \leq 3 \\ \frac{(x-3)(x+3)}{-(3-x)} = x+3 & \text{if } 3-x < 0, x > 3 \end{cases}$

$\left.\begin{array}{l} (LL) \lim_{x\to 3^-} -(x+3) = -6 \\ (RL) \lim_{x\to 3^+} (x+3) = 6 \end{array}\right\}$ $LL \neq RL \Rightarrow \lim_{x\to 3} f(x)$ $\boxed{\text{DNE}}$

5. Find the value(s) of a, if any, so that the following function is continuous at $x = a$. Make sure to use the definition of a function continuous at a number $x = a$.

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

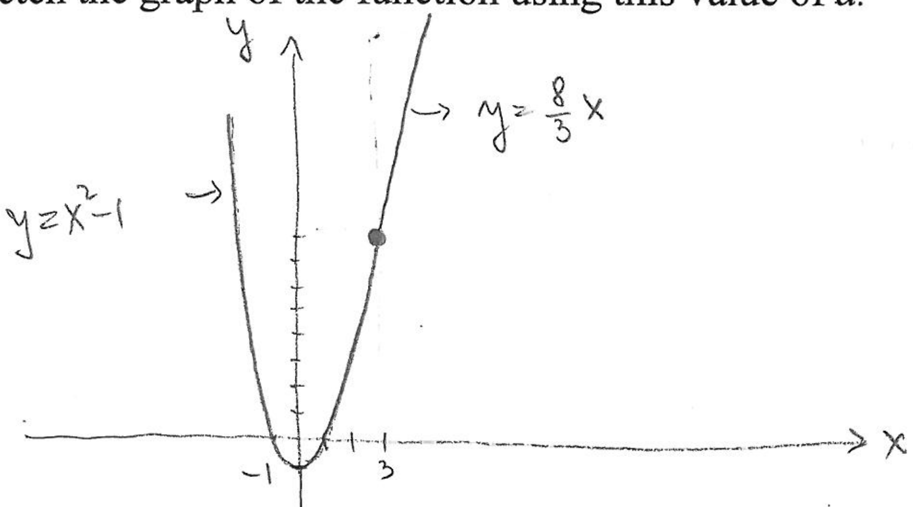
$\lim_{x \to 3^-} (x^2 - 1) = 9 - 1 = 8$

$f(3) = \lim_{x \to 3^+} 2ax = 6a$

$\Rightarrow 6a = 8 \Rightarrow a = \frac{4}{3}$

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ \frac{8}{3}x, & x \geq 3 \end{cases}$$

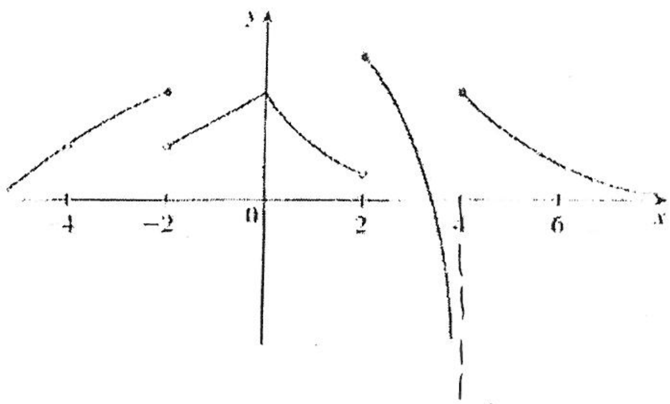
Sketch the graph of the function using this value of a.



Is the function continuous everywhere? Explain why or why not.

Yes
- each piece is cont. on $(-\infty, 3)$ and $[3, \infty)$ respectively because each piece is a polynomial
- at $x = 3$ it is cont. because this is how we determined the value of $a = \frac{4}{3}$

6. At what numbers x is the following function discontinuous? Identify the type of discontinuity.



x=4
VA

x	Type of discontinuity
-4	removable
-2	jump
2	jump
4	infinite

Write the equation(s) of the vertical asymptote(s), if any.

x=4

7. What is the half-life of a radioactive substance if 30% of it decays in 15 years?

Let $W(t)$ = the amount of substance at time t (years).

Then $W(t) = W_0 e^{-\lambda t}$

$$\left.\begin{aligned} W(15) &= 0.7 W_0 \\ W(15) &= W_0 e^{-15\lambda} \end{aligned}\right\} \Rightarrow 0.7 W_0 = W_0 e^{-15\lambda}$$

$$0.7 = e^{-15\lambda} \Rightarrow \ln(0.7) = -15\lambda \Rightarrow \lambda = -\frac{\ln(0.7)}{15}$$

$$T_h = \frac{\ln(2)}{\frac{-\ln(0.7)}{15}} = -\frac{15\ln(2)}{\ln(0.7)} \approx \boxed{29 \text{ years}}$$