MAT 2331 C
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
Final Exam
April 27, 2005

Instructor: Dr. Steve Desjardins

Duration: 3 hours

Name:____

Student Number:____________________________

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 11 pages.
- You must answer all questions. There are 8 questions worth a total of 60 marks.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary. Use the backs of the formula pages for rough work if needed.
- There are 2 pages of formulas (which may be detached) at the end of the exam.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

Question 1 (7 marks)
Solve the initial value problem:

$$(6xy^2 - 3x^2y^3 + 2y)\,dx + (9x^2y - 4x^3y^2 + 4x)\,dy = 0, \qquad y(1) = 1\,.$$

Question 2 (8 marks)
Solve the initial value problem:

$$y''' - 2y'' - 3y' = 6 + 10\cos x, \qquad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = 0\,.$$

Question 3 (8 marks)
Solve the initial value problem:

$$y'' - 2y' + y = \sqrt{x}\, e^x, \qquad y(0) = 2, \quad y'(0) = 1\,.$$

Question 4 (8 marks)
Solve the nonhomogeneous system:

$$\begin{aligned} y_1' &= 3y_1 - y_2 + 24t^2 - 16t \\ y_2' &= y_1 + y_2 + 12 \end{aligned}$$

Question 5 (8 marks)
Find the following:

a) $\mathcal{L}\{te^{-2t}\sin(3t)\}$

b) $\mathcal{L}\{u(t-2)\,(t^2-7t+3)\}$

c) $\mathcal{L}^{-1}\left\{\dfrac{s-16}{s^2-2s-8}\right\}$

d) $\mathcal{L}^{-1}\left\{\dfrac{2\,e^{-\pi s}}{s+3}\right\}$

Question 6 (7 marks)
Use the Laplace Transform to solve the initial value problem:

$$y'' - 4y' - 5y = \delta(t-3), \qquad y(0) = 4, \quad y'(0) = 2\,.$$

Question 7 (7 marks)
Consider the data points $\{(x_i, f(x_i))\}$: $(0.5, 1.2)$, $(1.1, 2.6)$ and $(1.9, 4.1)$.
a) Using Newton's Divided Differences, find the interpolating polynomial (with coefficients to 4 decimal places) that passes through the points. Verify that your $p_n(x)$ does pass through the points.
b) Interpolate a value at $x = 1.5$.
c) If $0.75 \leq |f'''(x)| \leq 2$ on $[0.5, 1.9]$, find error bounds for the interpolation.

Question 8 (7 marks)
Use the Runge-Kutta Method of order 4 to find y_1 and y_2 to 6 decimal places to approximate the solution of $y' = 3x^2y$, $y(0) = 2$, with $h = 0.5$. Compare the approximations with the true values by calculating the errors.

Formulas

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
t^n	$n!/s^{n+1}$	; $n = 0, 1, 2, \ldots$ and $s > 0$
e^{at}	$1/(s-a)$	; $s > a$
$\sin(kt)$	$k/(s^2+k^2)$	; $s > 0$
$\cos(kt)$	$s/(s^2+k^2)$	; $s > 0$
$\sinh(kt)$	$k/(s^2-k^2)$	; $s > k$
$\cosh(kt)$	$s/(s^2-k^2)$	; $s > k$
$\delta(t-a)$	e^{-as}	; $s > 0$
$u(t-a)$	$\frac{e^{-as}}{s}$	; $s > 0$

$$
\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st} f(t)dt \\
\mathcal{L}\{e^{at}f(t)\} &= F(s-a) \\
\mathcal{L}\{u(t-a)f(t-a)\} &= e^{-as}F(s) \\
\mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^n}{ds^n} F(s) \\
\mathcal{L}\left\{\int_0^t f(x)dx\right\} &= \frac{1}{s}F(s) \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty F(x)dx \\
\mathcal{L}\left\{\frac{d^n}{dt^n} f(t)\right\} &= s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \\
(f*g)(t) &= \int_0^t f(x)g(t-x)dx \\
\mathcal{L}\{(f*g)(t)\} &= F(s)\,G(s)
\end{aligned}
$$

$$
\begin{aligned}
p_1(x) &= f(x_0) + (x-x_0)\,f[x_0,x_1] \\
p_2(x) &= f(x_0) + (x-x_0)\,f[x_0,x_1] + (x-x_0)\,(x-x_1)\,f[x_0,x_1,x_2] \\
p_3(x) &= f(x_0) + (x-x_0)\,f[x_0,x_1] + (x-x_0)\,(x-x_1)\,f[x_0,x_1,x_2] \\
&+ (x-x_0)\,(x-x_1)\,(x-x_2)\,f[x_0,x_1,x_2,x_3] \\
|\,f(x) - p_n(x)\,| &= |\,\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\,(x-x_0)\,(x-x_1)\,\cdots\,(x-x_n)\,| \\
& \quad \textit{where} \;\; x_0 \le \xi(x) \le x_n \\
f[x_j, x_{j+1}, x_{j+2}, \ldots, x_k] &= \frac{f[x_{j+1},\ldots,x_k] - f[x_j, x_{j+1}, \ldots, x_{k-1}]}{x_k - x_j}
\end{aligned}
$$

$$\int_a^b f(x)dx = h\sum_{j=1}^{n} f(x_j^*)\,, \qquad |\epsilon| \le \frac{1}{24}\,M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\;|f''(x)|$$

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{j=1}^{n}\left(f(x_{j-1}) + f(x_j)\right), \qquad |\epsilon| \le \frac{1}{12}\,M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\;|f''(x)|$$

$$\int_a^b f(x)dx = \frac{h}{3}\sum_{j=0}^{n-1}\left(f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})\right), \qquad |\epsilon| \le \frac{1}{180}\,M\,(b-a)\,h^4\,,$$

$$M = max_{a\le x\le b}\;|f^{(4)}(x)|$$

$$y_{n+1}^P = y_n^C + h\,f(x_n, y_n^C)$$

$$y_{n+1}^C = y_n^C + \frac{1}{2}h\left(f(x_n, y_n^C) + f(x_{n+1}, y_{n+1}^P)\right)$$

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f(x_n + \frac{1}{2}\,h, y_n + \frac{1}{2}\,k_1)$$

$$k_3 = h\,f(x_n + \frac{1}{2}\,h, y_n + \frac{1}{2}\,k_2)$$

$$k_4 = h\,f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4)$$

Order n	Nodes t_i	Coefficients A_i
2	-0.5773502692	1.0
	0.5773502692	1.0
3	-0.7745966692	0.555555556
	0.0	0.888888889
	0.7745966692	0.555555556
4	-0.8611363116	0.3478548451
	-0.3399810436	0.6521451549
	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	-0.9061798459	0.2369268850
	-0.5384693101	0.4786286705
	0.0	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850