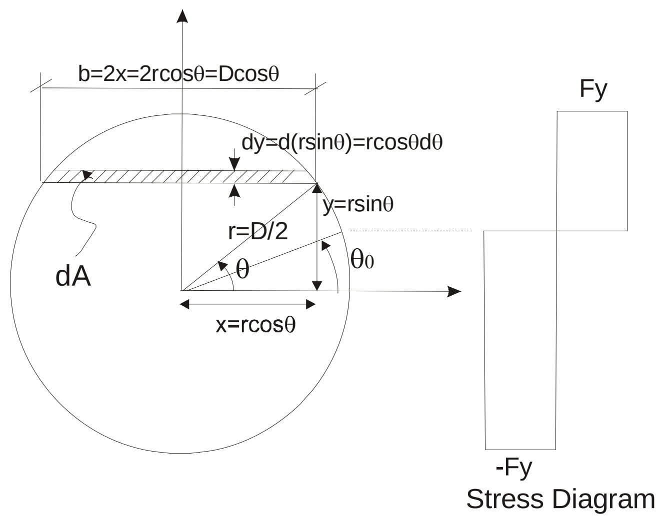
CVG 4143-STRUCTURAL STEEL II
ASSIGNMENT 2-Beam Columns

Plastic Interaction Relations for a solid circular section of diameter D



Element of Area $dA = bdy = (D\cos\theta)d\left(\frac{D}{2}\sin\theta\right) = \left(\frac{D^2}{2}\cos^2\theta\right)d\theta$

The element of area dA has to sweep the whole cross-section. This is accomplished by varying θ from $-\pi/2$ to $+\pi/2$

Stress distribution

$\sigma = -F_y \quad -\pi/2 \le \theta \le \theta_0$

$\quad +F_y \quad \theta_0 \le \theta \le +\pi/2$

The tensile axial force is the summation (or integration) of the longitudinal stresses over the cross-section.

$$P = \int_A \sigma dA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \sigma\left(\frac{D^2}{2}\cos^2\theta\right)d\theta = -\int_{\theta=-\frac{\pi}{2}}^{\theta=\theta_0} F_y \frac{D^2}{2}\left(\cos^2\theta\right)d\theta + \int_{\theta=\theta_0}^{\theta=\frac{\pi}{2}} F_y \frac{D^2}{2}\left(\cos^2\theta\right)d\theta \qquad (1)$$

<u>Note:</u>

$$\int \cos^2\theta d\theta = \int \frac{1}{2}(1+\cos 2\theta)d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C \qquad (2)$$

From Eq. 2, by substituting into Eq. 1., we get

$$P = -F_y \frac{D^2}{2}\left[\theta_0 + \frac{\sin 2\theta_0}{2}\right] = -F_y \frac{D^2}{2}\left[\theta_0 + \sin\theta_0 \cos\theta_0\right] \quad (3)$$

The bending moment is the summation of the moments due to the longitudinal stresses over the cross-section.

$$M = \int_A \sigma y dA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \sigma \left(\frac{D^2}{2}\cos^2\theta\right)\underbrace{\left(\frac{D}{2}\sin\theta\right)}_{moment\ arm} d\theta$$
$$= -\int_{\theta=-\frac{\pi}{2}}^{\theta=\theta_0} F_y \left(\frac{D^2}{2}\cos^2\theta\right)\left(\frac{D}{2}\sin\theta\right)d\theta + \int_{\theta=\theta_0}^{\theta=\frac{\pi}{2}} F_y \left(\frac{D^2}{2}\cos^2\theta\right)\left(\frac{D}{2}\sin\theta\right)d\theta \quad (4)$$

We note since the stress σ has a stepwise definition, the integral in Eq. has to be split into two integrals.

Note:

$$\int \cos^2\theta \sin\theta d\theta = -\frac{\cos^3\theta}{3} + C \quad (5)$$

From Eq. 5, by substituting into Eq. 4., we get

$$M = \frac{1}{6}F_y D^3 \cos^3\theta_0 \quad (6)$$

From Eq. 6,

$$\cos\theta_0 = \left(\frac{6M}{F_y D^3}\right)^{1/3}, \ \theta_0 = \cos^{-1}\left(\frac{6M}{F_y D^3}\right)^{1/3},$$
$$\sin\theta_0 = \sqrt{1-\cos^2\theta_0} = \sqrt{1-\left(\frac{6M}{F_y D^3}\right)^{2/3}} \quad (7)$$

From Eqs. (7), substitute into Eq. 3 to obtain the interaction equation:

$$P = \frac{F_y D^2}{2}\left[\cos^{-1}\left(\frac{6M}{F_y D^3}\right)^{1/3} + \left(\frac{6M}{F_y D^3}\right)^{1/3}\sqrt{1-\left(\frac{6M}{F_y D^3}\right)^{2/3}}\right] \quad (8)$$

Limiting Cases:

1. When $\theta_0 = \pm\frac{\pi}{2}$, the stress distribution corresponds to the case of pure axial force. Equations 3 and 6 yield the expressions $P = \pm\underbrace{\frac{\pi D^2}{2}}_{area}F_y = \pm AF_y = \pm P_y$ (maximum axial yield resistance), and $M = 0$ (zero bending resistance since the section is fully utilized in providing the maximum axial resistance).
2. When $\theta_0 = 0$, $M = \frac{D^3}{6}F_y = \underbrace{Z}_{plastic\ \mathrm{mod}ulus} F_y = M_p$ (maximum bending resistance) and $P = 0$ (no axial resistance since the section is fully utilized in providing the bending resistance)

Normalization:

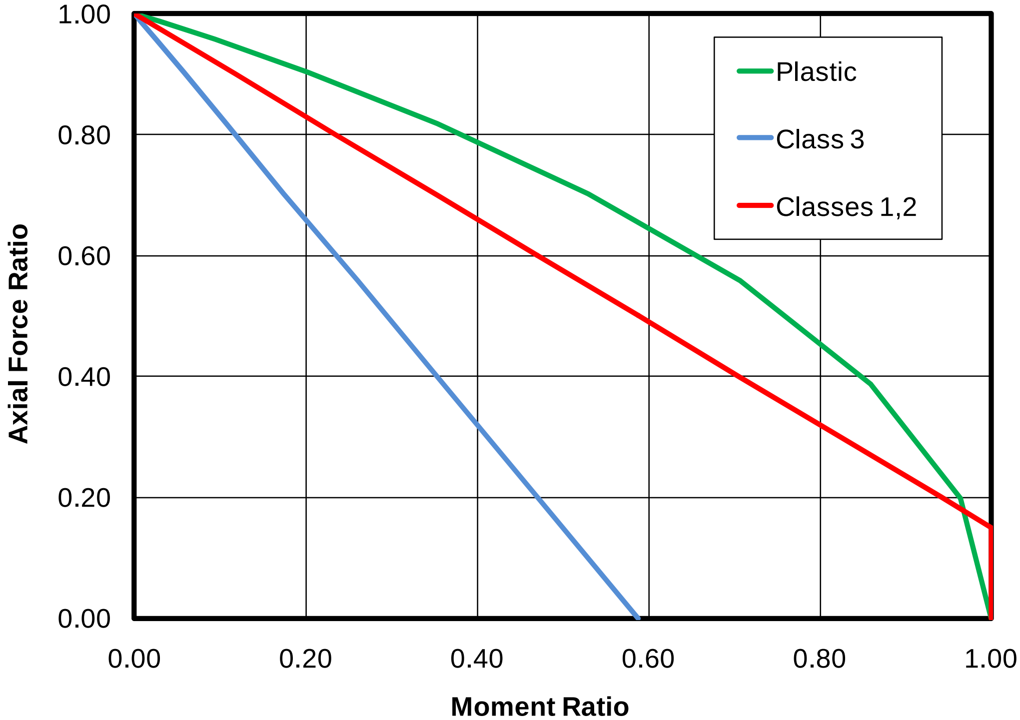
Divide both sides of Eq. 8 by P_y, we obtain

$$\frac{P}{P_y} = \frac{\frac{F_y D^2}{2}}{P_y}\left[\cos^{-1}\left(\frac{6M}{F_y D^3}\right)^{1/3} + \left(\frac{6M}{F_y D^3}\right)^{1/3}\sqrt{1-\left(\frac{6M}{F_y D^3}\right)^{2/3}}\right] \quad (9)$$

By using the expressions $P_y = AF_y$ and $M_p = \frac{D^3}{6}F_y$ derived under limiting cases, we obtain the following non-dimensional form of the interaction equation.

$$\frac{P}{P_y} = \pm\frac{2}{\pi}\left[\cos^{-1}\left(\frac{M}{M_p}\right)^{1/3} + \left(\frac{M}{M_p}\right)^{1/3}\sqrt{1-\left(\frac{M}{M_p}\right)^{2/3}}\right] \quad (10)$$

The above equation can be plotted by assuming ratios M/M_p and determining the corresponding P/P_y



Plastic Interaction Relation for Solid Circular Section