MATH 3705* C Test 1 January 2008

SOLUTIONS

Questions 1-4 are multiple choice. Circle the correct answer. Only the answer will be marked. [12 marks]

1. **[3]** $\mathcal{L}\{e^{4t}\cos(5t)\} =$

$(a)\ \dfrac{s+4}{(s+4)^2+25}$ $(b)\ \dfrac{s-4}{(s-4)^2+25}$ $(c)\ \dfrac{s+4}{s^2+25}$ (d) None of the above

By the 1-st Shifting Theorem, $\mathcal{L}\{e^{4t}\cos(5t)\} = F(s-4)$, where

$$F(s) = \mathcal{L}\{\cos(5t)\} = \frac{s}{s^2+5^2} \Rightarrow F(s-4) = \frac{s-4}{(s-4)^2+25}$$

.

2. **[3]** $\mathcal{L}\{u(t+1)\, e^{3t}\} =$

$(a)\ \dfrac{e^s}{s-3}$ $(b)\ \dfrac{e^{3-s}}{s-3}$ $(c)\ \dfrac{e^{s-3}}{s-3}$ (d) None of the above

By the 2-nd Shifting Theorem, $\mathcal{L}\{u(t+1)\, e^{3t}\} = \mathcal{L}\{u(t+1)\, e^{3(t+1)-3}\} =$

$$e^{-3}\mathcal{L}\{u(t+1)\, e^{3(t+1)}\} = e^{-3}\, e^{1\cdot s}\mathcal{L}\{e^{3t}\} = e^{s-3}\,\frac{1}{s-3}$$

.

3. [**3**] $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2-25}\right\} =$

(a) $e^{5t} - e^{-5t}$ (b) $e^{5t} + e^{-5t}$ (c) $-\cos(5t)$ (d) None of the above

$$\frac{2s}{s^2-25} = \frac{2s}{(s-5)(s+5)} = \frac{A}{s-5} + \frac{B}{s+5} = \frac{A(s+5)+B(s-5)}{(s-5)(s+5)} = \frac{(A+B)s+(5A-5B)}{(s-5)(s+5)}.$$

Thus, $A-B=0$, so $A=B$. Since $A+B=2$, then $A=B=1$, and $\dfrac{2s}{s^2-25} = \dfrac{1}{s-5} + \dfrac{1}{s+5}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s-5} + \frac{1}{s+5}\right\} = e^{5t} + e^{-5t}.$$

4. [**3**] $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4}\right\} =$

(a) $u(t-3)\ \cos(2t)$ (b) $u(t+3)\ \cos[2(t+3)]$

(c) $u(t-3)\ \cos[2(t-3)]$ (d) None of the above

By the 2-nd Shifting Theorem, $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4}\right\} = u(t-3)f(t-3)$, where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t)$$

.

5. [**4 marks**] Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} e^{2t}, & \text{if } 0 \,□\, t < 3; \\ 0, & \text{if } t \geq 3. \end{cases}$

Solution:

Rewrite $f(t)$ using the unit step function $u(t-a)$:

$$f(t) = [u(t) - u(t-3)]e^{2t} + u(t-3) \cdot 0 = u(t)e^{2t} - u(t-3)e^{2t}.$$

Find the Laplace transform of each term of the expression above, using the Second Shifting theorem with $a = 0$ and $a = 3$ respectively:

$$\mathcal{L}\{u(t)e^{2t}\} = e^{0 \cdot s}\mathcal{L}\{e^{2t}\} = \frac{1}{s-2};$$

$$\mathcal{L}\{u(t-3)e^{2t}\} = \mathcal{L}\{u(t-3)e^{2(t-3)+6}\} = e^{6}e^{-3s}\mathcal{L}\{e^{2t}\} = e^{(6-3s)}\frac{1}{s-2}.$$

Answer: $\dfrac{1}{s-2} - e^{(6-3s)}\dfrac{1}{s-2} = \dfrac{1 - e^{(6-3s)}}{s-2}.$

6. [**7 marks**] Employ the Laplace transform to solve the initial-value problem

$$y'' + 2y' + 10y = 0, \quad y(0) = 1, \; y'(0) = 2$$

.

Solution:

Take the Laplace transform of both parts of the equation:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 10Y(s) = 0.$$

Substitute $y(0) = 1, \; y'(0) = 2$ and solve for $Y(s)$:

$$(s^2 + 2s + 10)Y(s) = s + 4, \; \Rightarrow \; Y(s) = \frac{s+4}{s^2 + 2s + 10} = \frac{(s+1)+3}{(s+1)^2 + 9}.$$

Using the First Shifting Theorem with $a = -1$, find

$$\mathcal{L}^{-1}\{Y(s)\} = e^{-t}\cos(3t) + e^{-t}\sin(3t) = y(t).$$

7. [**7 marks**] Find $f(t)$, if

$$f(t) - 2\int_0^t f(x)\,dx = 2t.$$

Solution:

Let $\mathcal{L}\{f(t)\} = F(s)$. Take the Laplace transform of both parts of the equation:

$$F(s) - 2\frac{1}{s}F(s) = 2\frac{1}{s^2} \quad \Rightarrow F(s)\left(1 - \frac{2}{s}\right) = 2\frac{1}{s^2} \quad \Rightarrow F(s) = \frac{2}{s(s-2).}$$

To invert $F(s)$, one can use (a) partial fractions or (b) $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{F(s)}{s}$.

(a) $F(s) = \dfrac{2}{s(s-2)} = -\dfrac{1}{s} + \dfrac{1}{s-2} \quad \Rightarrow \quad y(t) = -1 + e^{2t}.$

OR

(b) $\dfrac{2}{s(s-2)} = \dfrac{1}{s} \cdot \dfrac{2}{s-2} = \dfrac{1}{s} \cdot F(s)$. Then $\mathcal{L}^{-1}\Big\{F(s)\Big\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s-2}\right\} = 2e^2t = f(t),$
and

$$y(t) = \int_0^t f(x)\,dx = 2\int_0^t e^{2x}\,dx = 2 \cdot \frac{1}{2}e^{2x}\Big|_0^t = e^{2t} - 1.$$