MAT 2384 - Solution to Assignment #4

Question 1 1) $x^2y''+xy'+4y=0$, $x>0$

This is an Euler-Cauchy Equation with $a=1$, $b=4$.

Its characteristic equation is $m^2+(a-1)m+b=0 \Rightarrow$

$m^2+0m+4=0 \iff m=\pm 2i$; 2 Complex Conjugate roots.

The general solution is $y(x)=C_1\cos(2\ln x)+C_2\sin(2\ln x)$

2) $x^2y''-6y=0$, $x>0$, $y(1)=5$, $y'(1)=5$

This is an Euler-Cauchy Equation with $a=0$, $b=-6$.

The characteristic equation is $m^2+(0-1)m-6=0 \iff$

$m^2-m-6=0 \Rightarrow (m+2)(m-3)=0 \Rightarrow m_1=-2, m_2=3$;

2 distinct real roots. The general solution is then

$$y(x)=C_1x^{-2}+C_2x^3$$

$$y'(x)=-2C_1x^{-3}+3C_2x^2$$

$$\left.\begin{array}{l} y(1)=5 \Rightarrow C_1+C_2=5 \\ y'(1)=5 \Rightarrow -2C_1+3C_2=5 \end{array}\right\} \Rightarrow \begin{array}{l} 5C_2=15 \Rightarrow C_2=3 \\ C_1=2 \end{array}$$

The unique solution is $y(x)=2x^{-2}+3x^3$

3) $x^2y''+5xy'+4y=0$, $x>0$, $y(1)=5$, $y'(1)=-9$

This is an Euler-Cauchy Equation with $a=5$, $b=4$.

Its characteristic equation is $m^2-4m+4=0 \iff (m-2)^2=0$

$\Rightarrow m_1 = m_2 = 2$. The general solution in this case is

$$y(x) = c_1 x^2 + c_2 x^2 \ln x$$

$y'(x) = 2c_1 x + 2c_2 x \ln x + c_2 x$; $y'(1) = -9 \Rightarrow$

$2c_1 + c_2 = -9$; $y(1) = 5 \Rightarrow c_1 = 5$. So $c_1 = 5$, $c_2 = -19$.

The unique solution is $y(x) = 5x^2 - 19x^2 \ln x$

4) $y''' - 5y'' + 2y' + 8y = 0$, $y(0) = 2$, $y'(0) = -1$, $y''(0) = -5$.

The characteristic equation is $\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$ which can be written as $(\lambda+1)(\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = -1, \lambda = 2, \lambda = 4$; three distinct real roots. The general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{4x}$$

$y'(x) = -c_1 e^{-x} + 2c_2 e^{2x} + 4c_3 e^{4x}$; $y'(0) = -1 \Rightarrow -c_1 + 2c_2 + 4c_3 = -1$

$y''(x) = c_1 e^{-x} + 4c_2 e^{2x} + 16c_3 e^{4x}$; $y''(0) = -5 \Rightarrow c_1 + 4c_2 + 16c_3 = -5$

$y(0) = 2 \Rightarrow c_1 + c_2 + c_3 = 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & 2 & 4 & -1 \\ 1 & 4 & 16 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 5 & 1 \\ 0 & 3 & 15 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 10 & -8 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 5/3 & 1/3 \\ 0 & 0 & 1 & -\frac{4}{5} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{14}{5} \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -4/5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17/15 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & -4/5 \end{array}\right]$$

$\Rightarrow c_1 = 17/15$, $c_2 = 5/3$, $c_3 = -4/5$.

The unique solution is $y(x) = \frac{17}{15} e^{-x} + \frac{5}{3} e^{2x} - \frac{4}{5} e^{4x}$

5) $y''' + 9y'' + 27y' + 27y = 0$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = 3$

The characteristic equation is $\lambda^3 + 9\lambda^2 + 27\lambda + 27 = 0$.

By inspection, $\lambda = -3$ is a root. Performing the long division with $\lambda + 3$, we get $\lambda^3 + 9\lambda^2 + 27\lambda + 27 =$

$$(\lambda+3)(\lambda^2 + 6\lambda + 9) = (\lambda+3)^3.$$ Therefore, the characteristic equation has a triple root $\lambda = -3$. The general solution is

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} + c_3 x^2 e^{-3x}.$$

$$y'(x) = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x} + 2c_3 x e^{-3x} - 3c_3 x^2 e^{-3x}$$

$$y'(0) = 0 \Rightarrow -3c_1 + c_2 = 0 \quad (1)$$

$$y''(x) = 9c_1 e^{-3x} - 3c_2 e^{-3x} - 3c_2 e^{-3x} + 9c_2 x e^{-3x} + 2c_3 e^{-3x} - 6c_3 x e^{-3x}$$
$$- 6c_3 x e^{-3x} + 9c_3 x^2 e^{-3x}$$

$$y''(0) = 3 \Rightarrow 9c_1 - 6c_2 + 2c_3 = 3 \quad (2)$$

$$y(0) = 2 \Rightarrow c_1 + c_2(0)e^0 + c_3 0^2 e^0 = 2 \Rightarrow c_1 = 2$$

$(1) \Rightarrow c_2 = 3c_1 = 6$ and $(2) \Rightarrow c_3 = \frac{1}{2}[3 - 9c_1 + 6c_2] = \frac{21}{2}$

The unique solution is $y(x) = 2e^{-3x} + 6xe^{-3x} + 21x^2 e^{-3x}$

6) $y''' + 5y'' + 8y' + 4y = 0$, $y(0) = -1$, $y'(0) = 0$, $y''(0) = -1$

The characteristic equation is $\lambda^3 + 5\lambda^2 + 8\lambda + 4 = 0$.

By inspection, $\lambda = -1$ est une solution. En effectuant la division longue par $\lambda + 1$, on trouve

$$\lambda^3 + 5\lambda^2 + 8\lambda + 4 = (\lambda+1)(\lambda^2 + 4\lambda + 4) = (\lambda+1)(\lambda+2)^2$$

$\Rightarrow \lambda = -1$ is a simple real root and $\lambda = -2$ is a double root. The general solution is $y(x) = C_1 e^{-x} + C_2 e^{-2x} + C_3 x e^{-2x}$.

$y'(x) = -C_1 e^{-x} - 2C_2 e^{-2x} + C_3 e^{-2x} - 2C_3 x e^{-2x}$

$y'(0) = 0 \Rightarrow -C_1 - 2C_2 + C_3 = 0$ ①

$y''(x) = C_1 e^{-x} + 4C_2 e^{-2x} - 2C_3 e^{-2x} - 2C_3 e^{-2x} + 4C_3 x e^{-2x}$

$y''(0) = -1 \Rightarrow C_1 + 4C_2 - 4C_3 = -1$ ②

$y(0) = -1 \Rightarrow C_1 + C_2 = -1$ ③

$4$① $+$ ② $\Rightarrow -3C_1 - 4C_2 = -1$ ④

$3$③ $+$ ④ $\Rightarrow -C_2 = -4 \Rightarrow C_2 = 4$

③ $\Rightarrow C_1 = -1 - C_2 = -5$

① $\Rightarrow C_3 = C_1 + 2C_2 = -5 + 8 = 3$

The unique solution is $y(x) = -5e^{-x} + 4e^{-2x} + 3xe^{-2x}$

7) $y''' + y'' + 2y' + 2y = 0$. The characteristic equation is

$\lambda^3 + \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda^2(\lambda+1) + 2(\lambda+1) = 0 \Rightarrow (\lambda+1)(\lambda^2+2) = 0$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = -\sqrt{2}\,i,\ \lambda_3 = \sqrt{2}\,i$. So we have one simple real root and two complex conjugate roots. The general solution would then be

$y(x) = C_1 e^{-x} + C_2 \cos(\sqrt{2}x) + C_3 \sin(\sqrt{2}x)$

Note that e^{-x}, $\cos(\sqrt{2}x)$, $\sin(\sqrt{2}x)$ are linearly independent (one could verify this with the Wronskian) solutions.

Question 2 $f(0)=0$, $f(1)=-1.2546$, $f(2)=1.1478$ and $f(3)=2.2541$. Note that in this case, the nodes are equidistant with $h=1$. For $x=2.5$, we have that

$$r = \frac{x-x_0}{h} = \frac{2.5-0}{1} = 2.5.$$

Gregory-Newton Formula at $r=2.5$ gives:

$$\sum_{k=0}^{3} \binom{2.5}{k} \Delta^k f_0 = \binom{2.5}{0}\Delta^0 f_0 + \binom{2.5}{1}\Delta f_0 + \binom{2.5}{2}\Delta^2 f_0 + \binom{2.5}{3}\Delta^3 f_0$$

$\Delta f_0 = f_1 - f_0 = -1.2546 - 0 = -1.2546$

$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 = 3.657$

$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = \Delta f_2 - \Delta f_1 - 3.657$

$= f_3 - f_2 - (f_2 - f_1) - 3.657 = f_3 - 2f_2 + f_1 - 3.657$

$= 2.2541 - 2(1.1478) - 1.2546 - 3.657 = -4.9531$

$\binom{2.5}{0} = 1$, $\binom{2.5}{1} = \frac{2.5}{1!} = 2.5$, $\binom{2.5}{2} = \frac{2.5(2.5-1)}{2!} = 1.875$

$\binom{2.5}{3} = \frac{2.5(2.5-1)(2.5-2)}{3!} = 0.3125$

So $f(2.5) \simeq \sum_{k=0}^{3} \binom{2.5}{k} \Delta^k f_0 = 1.(0) + 2.5(-1.2546) +$

$1.875(3.657) + 0.3125(-4.9531) = 2.17253$